Math 431

We begin Math 431 by considering problems concerning the set, \( \mathbb{R} \), of real numbers. These numbers we consider to be in one-to-one correspondence with points on a line, ordered from left to right in the usual way. We will assume all the familiar arithmetic and order and thus, subsets of the numbers may be defined by statements involving arithmetic and order. For example:

\[ S = \{ x: x \text{ is in } \mathbb{R} \text{ and } 2 + x > 4 \} \]

is the set to which \( x \) belongs if and only if \( x \) is greater than 2.

**Def.** The number set, \( A \), is a right ray means that if \( x \) is in \( A \) and \( y > x \), then \( y \) is in \( A \). We similarly define a left ray.

In addition, we will assume that the following statement holds for the numbers. We call this statement an Axiom and emphasize that it is an assumption that does not follow from the usual properties of arithmetic and order. Nevertheless, it is a reasonable property for the points on a line to have and we wish to assume that the set of real numbers also has it.

**Axiom 1** If \( \mathbb{R} \) is the union of the nonempty left ray \( A \) and the nonempty right ray \( B \), and \( A \) and \( B \) do not intersect, then either \( A \) has a largest element or \( B \) has a smallest element.

If each of \( a \) and \( b \) is a number such that \( a < b \), then we will use the following notation and terminology.

- **open interval** \( (a,b) = \{ x: x \text{ is a number and } a < x \text{ and } x < b \} \)
- **half-open interval** \( [a,b)=\{x: x \text{ is a number and } a \leq x \text{ and } x < b \} \)
- **half-open interval** \( (a,b]=\{x: x \text{ is a number and } a < x \text{ and } x \leq b \} \)
- **closed interval** \( [a,b]=\{x: x \text{ is a number and } a \leq x \text{ and } x \leq b \} \)

**Def.** The statement that the number, \( p \), is a limit point of the number set, \( A \), means that if \( (a,b) \) is an open interval containing \( p \), (that is: \( a < p < b \)) then there is a number \( q \) such that \( a < q < b \), \( q \) is in \( A \), and \( q \) does not equal \( p \).

A more concise statement of the previous definition may read: \( P \) is a limit point of \( A \) if and only if each open interval containing \( p \) contains a number in \( A \) different from \( p \).

**Def.** The statement that the number set, \( A \), is closed means that if \( p \) is a limit point of \( A \) then \( p \) is in \( A \).

**Def.** The statement that the number set, \( A \), is open means that if \( p \) is in \( A \) then there is an open interval containing \( p \) that is contained in \( A \).

**Def.** The statement that \( f \) is a function means that \( f \) is a collection, each member of which is an ordered pair, no two of which have the same first coordinate. The set of first coordinates for \( f \) is called the domain of \( f \), while the set of second coordinates is called the image of \( f \).

**Def.** The statement that \( S \) is a sequence means that \( S \) is a function with domain some initial segment of the positive integers. (That is: the domain of \( S \) is either the set of positive integers or the domain of \( f \) is the set \( \{1,2,3,\ldots,n\} \) for some positive integer \( n \).)
Def. The statement that p is the limit of the sequence S means that if (a,b) is an interval containing p, then there is a positive integer N such that S(i) is in (a,b) for each positive integer i ≥ N.

Def. The statement that T is a subsequence of the sequence S means there is an increasing sequence, I, of positive integers such that T = S(I).

Def. The statement that the function \( f : A \rightarrow B \) is surjective (another name is “onto”) means that if y is in B then there is an x in A such that f(x) = y.

Def. The statement that the set A is countable means there is a surjective \( f : \mathbb{Z}^+ \rightarrow A \).

Problems

1. Find the limit point(s) of the following set: C = \{n*(2.5)mod 1 : n is a positive integer\}
2. There exists a sequence S such that S has a limit and \( \text{Im}(S) \) has exactly 2 limit points. (A direct proof)
3. If the sequence S has a limit then the limit is unique.
4. l.p. iff l.p.2 (l.p. is the open interval definition, l.p.2 uses distance)
5. There is a number set, A, which has the property that A contains no open interval and each point in A is a limit point of A.
6. If A is an infinite subset of [0,1] then A has a limit point.
7. If A is open and p is in A and there is a q > p such that q is not in A then there is a b not in A such that (p,b) is a subset of A.

Def. The statement that the number set A has measure zero means that if c > 0 there is a countable (possibly finite) collection of open intervals \( \left( a_i, b_i \right) \) such that \( A \subseteq \bigcup_i \left( a_i, b_i \right) \) and

8. If A is countable then A has measure 0.

Axiom 2 If A is a number set and b is a number such that if \( x \in A \) then \( x \leq b \), then there is a number q such that if \( x \in A \) then \( x \leq q \) and if \( r < q \) then there is a y in A such that \( r < y \leq q \).

10. If S is a monotonically non-decreasing number sequence which is bounded above, then S has a limit.

Def. If [a,b] is a closed interval, then by a subdivision of [a,b] we mean a finite collection D of non-overlapping closed intervals whose union is [a,b]. (Two closed intervals are nonoverlapping provided either they do not intersect or their intersection is a single point.) An equivalent definition is that a subdivision of [a,b] is a finite increasing sequence in [a,b] such that its initial point is a and is final point is b.

Def. The statement that the subdivision D refines the subdivision D’ means that if d is in D then there is an e in D’ such that d is a subset of e. In this case we say that D is a refinement of D’. (Equivalently using the sequence definition, D refines D’ means that D’ is a subsequence of D.)

Def. If D is a collection of sets, then a choice function for D is a function ch from D into the union of D with the property that ch(d) is in d for each d in D.
Def If $D$ is a subdivision of $[a,b]$, then the mesh of $D$ is $\max\{|d|: d \text{ is in } D\}$. ($|d|$ denotes the length of $d$.)

Def Suppose $[r,s]$ is a closed interval and $g$ is a real valued function whose domain contains $[r,s]$. the $g$-length of $[r,s]$ is the number $g(r)-g(s)$ and will be denoted by $g|[r,s]$.

We are now in a position to define two (possibly different) notions of integral. Suppose $I=[a,b]$ and each of $f$ and $g$ is a real valued function defined on $I$.

Def The statement that $f$ is integrable on $I$ with respect to $g$ means there is a number $m$ such that if $\varepsilon > 0$ there is a $\delta > 0$ such that if $D$ is a subdivision of $I$ with mesh less than $\delta$ and $ch$ is a choice function for $D$, then

$$\sum_{d \in D} f(ch(d)) \ g |d - m| < \varepsilon.$$  

In this case we will denote the number $m$ by $\int_a^b f \ dg$.

Def The statement that $f$ is type-R integrable on $I$ with respect to $g$ means there is a number $m$ such that if $\varepsilon > 0$ there is a subdivision $D'$ of $I$ such that if $D$ refines $D'$ and $ch$ is a choice function for $D$ then

$$\sum_{d \in D} f(ch(d)) \ g |d - m| < \varepsilon.$$  

In this case we will denote the number $m$ by $\int_a^b f \ dg$.

11. If $f$ is integrable on $I$ with respect to $g$ then the number $m$ is unique.

Def. If $p$ is a point in $\mathbb{R}^2$ and epsilon is greater than zero, then the epsilon ball centered at $p$ is defined to be the set of points within epsilon of $p$. This is also referred to as the epsilon neighborhood of $p$. This definition naturally extends the topology of the line to that of the plane, and beyond. Of course, since we also have limit points defined in terms of distance, as long as we have the concept of distance, which we already do in Euclidean n-space, it is possible to mimic many of the theorems from 431 to higher dimensional spaces.

12. If $A$ is a subset of $\mathbb{R}^2$ that is infinite and bounded, then $A$ has a limit point.
13. If $f$ is continuous over $[a,b]$ and $f(a) < y < f(b)$, then there is a $c$ in $D$ such that $y = f(c)$.
14. If $f$ is continuous over the compact set $A$ and $c > 0$, then there is a $d > 0$ such that if each of $x$ and $y$ is in $A$ and $|x - y| < d$, the $|f(x) - f(y)| < c$.
15. If $S$ is a sequence which satisfies if $c>0$ there is a positive integer $N$ such that if $i,j > N$ then $|S(i) - S(j)|<c$, then $S$ has a limit.
16. $\int_a^b fd (g + h) = \int_a^b f\ dg + \int_a^b f\ dh$.
17. If $f$ is continuous over $U$ and $V$ is a subset of $U$ then $f$ restricted to $V$ is continuous.
18. a. If $f[0,1] --> \mathbb{R}$ is non-decreasing, then $f$ is B.V.
18b. If $f : [0, 1] \rightarrow \mathbb{R}$ and $a \in (0, 1)$ and each of $f_{[0,a]}$ and $f_{[a,1]}$ is B.V. then $f$ is B.V.

19. Show $f(x) = \begin{cases} 0 & 0 \leq x < 0.5 \\ 1 & 0.5 \leq x \leq 1 \end{cases}$ is Riemann integrable over $[0, 1]$.

20. Show that every point in the Cantor set is a limit point of the Cantor set.

21. The product of continuous functions is continuous over a common domain.