We begin Math 431 by considering problems concerning the set, $\mathbb{R}$, of real numbers. These numbers we consider to be in one-to-one correspondence with points on a line, ordered from left to right in the usual way. We will assume all the familiar arithmetic and order and thus, subsets of the numbers may be defined by statements involving arithmetic and order. For example:

$$S = \{ x : x \text{ is in } \mathbb{R} \text{ and } 2 + x > 4 \}$$

is the set to which $x$ belongs if and only if $x$ is greater than 2.

**Def.** The number set, $A$, is a right ray means that if $x$ is in $A$ and $y > x$, then $y$ is in $A$.

We similarly define a left ray.

In addition, we will assume that the following statement holds for the numbers. We call this statement an Axiom and emphasize that it is an **assumption** that does not follow from the usual properties of arithmetic and order. Nevertheless, it is a reasonable property for the points on a line to have and we wish to assume that the set of real numbers also has it.

**Axiom 1** If $\mathbb{R}$ is the union of the nonempty left ray $A$ and the nonempty right ray $B$, and $A$ and $B$ do not intersect, then either $A$ has a largest element or $B$ has a smallest element.

If each of $a$ and $b$ is a number such that $a < b$, then we will use the following notation and terminology.

- **open interval** $$(a,b) = \{ x : x \text{ is a number and } a < x \text{ and } x < b \}$$
- **half-open interval** $$[a,b) = \{ x : x \text{ is a number and } a \leq x \text{ and } x < b \}$$
- **half-open interval** $$(a,b] = \{ x : x \text{ is a number and } a < x \text{ and } x \leq b \}$$
- **closed interval** $$[a,b] = \{ x : x \text{ is a number and } a \leq x \text{ and } x \leq b \}$$

**Def.** The statement that the number, $p$, is a limit point of the number set, $A$, means that if $(a,b)$ is an open interval containing $p$, (that is: $a < p < b$) then there is a number $q$ such that $a < q < b$, $q$ is in $A$, and $q$ does not equal $p$.

A more concise statement of the previous definition may read: $P$ is a limit point of $A$ if and only if each open interval containing $p$ contains a number in $A$ different from $p$.  

**Def.** The statement that the number set, $A$, is **closed** means that if $p$ is a limit point of $A$ then $p$ is in $A$.

**Def.** The statement that the number set, $A$, is **open** means that if $p$ is in $A$ then there is an open interval containing $p$ that is contained in $A$.

**Def.** The statement that $f$ is a **function** means that $f$ is a collection, each member of which is an ordered pair, no two of which have the same first coordinate. The set of first coordinates for $f$ is called the domain of $f$, while the set of second coordinates is called the image of $f$.

**Def.** The statement that $S$ is a **sequence** means that $S$ is a function with domain some initial segment of the positive integers. (That is: the domain of $S$ is either the set of positive integers or the domain of $f$ is the set $\{1,2,3,...,n\}$ for some positive integer $n$.)
**Def.** The statement that \( p \) is the limit of the sequence \( S \) means that if \((a,b)\) is an interval containing \( p \), then there is a positive integer \( N \) such that \( S(i) \) is in \((a,b)\) for each positive integer \( i \geq N \).

**Def.** The statement that \( T \) is a subsequence of the sequence \( S \) means there is an increasing sequence, \( I \), of positive integers such that \( T = S(I) \).

**Def.** The statement that the function \( f : A \to B \) is surjective (another name is “onto”) means that if \( y \) is in \( B \) then there is an \( x \) in \( A \) such that \( f(x) = y \).

**Def.** The statement that the set \( A \) is countable means there is a surjective \( f : \mathbb{Z}^+ \to A \).

Problems:

1. There is a number set, \( A \), such that \( 0 \) is a limit point of \( A \).
2. If \( A \) is a number set and \( A \) has a limit point, \( p \), then \( A \) is infinite.
3. If \( p \) is a limit point of \( A \), then \( p \) is in \( A \).
4. If \( A \) is infinite, then \( A \) has a limit point.
   4b. If \( A \) is an infinite subset of \([0,1]\) then \( A \) has a limit point.
   4c. If \( A \) is uncountable then \( A \) has a limit point.
5. Find a positive (without using the words none or no) statement that means that the set \( A \) is infinite. (i.e. not finite is not what we’re looking for)
6. The closed interval \([0,1]\) is infinite.
7. If \( c > 0 \) there is a positive integer \( N \) such that \( 1/N < c \).
8. There is a number set, \( A \), which has the property that \( A \) contains no open interval and each point in \( A \) is a limit point of \( A \).
9. There is a closed number set, \( B \), which satisfies the properties of \( A \) in problem 8.
10. There is a number, \( p \), such that \( p \times p = 2 \).
11. There is a number set that is neither open nor closed.
12. There is a number set that is both open and closed.
13. The rational numbers in \([0,1]\) form a countable set.
14. There is a set with exactly one limit point.
15. Find the limit point(s) of the following sets.
   \(A = \{1 + 1/2 + 1/3 + \ldots + 1/n : n \text{ is a positive integer}\}\)
   \(B = \{e(1)*1 + e(2)*1/2 + \ldots + e(n)*1/n : n \text{ is a positive integer and } e(i) \text{ is in } \{1,-1\} \text{ for each } i\}\)
   \(C = \{n*(2^{-5}) \text{mod 1} : n \text{ is a positive integer}\}\)

   Let \(a\) be a number, \(f\) be the real valued function defined by \(f(x) = x^2 - 2\), and define 
   \(S_a = \{f^n(a) : n \text{ is a positive integer}\}\). Here \(f^1(x) = f(x)\) and \(f^n(x) = f(f^{n-1}(x)) = f \circ f^{n-1}(x)\).

16. Assuming \(A\) is not \(\mathbb{R}\), \(A\) is open iff \(\mathbb{R} - A\) is closed.

   **Def.** The statement that the set \(B\) is the closure of the set \(A\) means that \(B\) is the union of \(A\) and the set of limit points of \(A\). (In the case \(A\) has no limit point, then the closure of \(A\) is simply \(A\).)

17. The closure of \(A\) is closed.

18. The statement that \(p\) is not a limit point of the set \(A\) means…

19. If \(p\) is a number, then there is a sequence \(S\) that has limit \(p\).

20. There exists a sequence \(S\) such that \(S\) has a limit and \(\text{Im}(S)\) has exactly 2 limit points.

   20a. If the sequence \(S\) has a limit then the limit is unique.
   20b. Can the \(\text{Im}(s)\) be uncountable?
   20c. Direct proof for number 20.


22. \(\text{l.p. if l.p.2 (l.p. is the open interval definition, l.p.2 uses distance)}\)

23. If each of \(A\) and \(B\) is countable, then \(A \cup B\) is countable.

   a) The countable union of countable sets is countable.

**Axiom 2** If \(A\) is a number set and \(b\) is a number such that if \(x \in A\) then \(x \leq b\), then there is a number \(q\) such that if \(x\) is in \(A\) then \(x \leq q\) and if \(r < q\) then there is a \(y\) in \(A\) such that \(r < y \leq q\).


25. If \(A\) is a collection of open number sets, then \(\bigcup_{x \in A} X\) is open.

26. If \(A\) is a collection of closed number sets and \(p\) is a number satisfying: if \(x\) is in \(A\) then \(p\) is in \(x\), then \(\bigcap_{x \in A} x\) is closed.

27. If \(p\) is a limit of the sequence \(S\), then \(p\) is a limit point of \(\text{Im}(S)\). (Shown to be false, but improved to \(\text{Im}(S)\) is infinite.)
The statement that the sequence \( \{A_i\} \) of number sets is \textit{monotonically non-increasing} means if \( i \) is a positive integer then \( A_i \subseteq A_{i+1} \). We similarly define \textit{monotonically non-decreasing}.

28. If \( \{A_i\} \) is a monotonically non-increasing sequence of intervals, then there is a \( p \) such that if \( i \) is a positive integer then \( p \) is in \( A_i \). Shown to be false, but true if intervals are closed.

\[ \text{Def.} \quad \text{T is a subsequence of the sequence S means there is an increasing sequence I of integers such that } T = S \circ I \]

29. If \( p \) is a limit point of \( \text{Im}(S) \), then there exists a subsequence \( T \) with limit \( p \).

30. \( .725123123123\ldots = ? \)

31. If \( A \) is open and \( p \) is in \( A \) and there is a \( q > p \) such that \( q \) is not in \( A \) then there is a \( b \) not in \( A \) such that \( (p,b) \) is a subset of \( A \).

32. If \( A \) is countable then \( A \) has measure 0.

33. If \( A \) is open then there exists a sequence of open intervals \( I \) such that \( A = \bigcup_j I_j \)

\[ \text{Def.} \quad \text{The number set } A \text{ has measure zero means that if } c > 0 \text{ there is a countable (possibly finite) collection of open intervals } (a_i,b_i) \text{ such that } A \subseteq \bigcup_i (a_i,b_i) \text{ and } \sum_i b_i - a_i < c. \]

34. \( T_1. \) Is \( \frac{1}{4} \) in the \( T \)-set? showed false if we use \( \pi / 4 \).

\[ \text{T}_2. \text{ Is } T \text{ countable?} \]

\[ \text{T}_3. \text{ Show that each point in } T \text{ is a limit point of } T. \]

\[ \text{T}_4. \text{ Show that } T \text{ contains no open interval.} \]

\[ \text{T}_5. \text{ Show that } T \text{ has measure } 0. \]

\[ \text{T}_6. \text{ Show there is a “fat” } T \text{-set.} \]

\[ \text{T}_7. \text{ Every “random” } T \text{-set is homeomorphic to the } T \text{-set.} \]

35. Suppose for each positive integer \( n \), \( G_n \) is an open set with the property that if \( x \) is a number then \( x \) is a limit point of \( G_n \). Let \( G = \bigcap_n G_n \). Show that each number is a limit point of \( G \). (i.e. If \( G \) is a sequence of open dense sets then the intersection of the members
36. Suppose for each integer \( n \), \( F_n \) is a closed set that contains no open interval, and \( F = \bigcup_{n} F_n \).

Show that \( F \) is not equal to the numbers.

37. \( [0,1] \) is uncountable.

38. If \( S \) is a monotonically non-decreasing number sequence which is bounded above, then \( S \) has a limit.

39. If \( S \) is a sequence which satisfies if \( c > 0 \) there is a positive integer \( N \) such that if \( i, j > N \) then \( |S(i) - S(j)| < c \), then \( S \) has a limit.

40. Suppose \( G \) is a collection of open intervals such that if \( x \) is in \( [0,1] \) then \( x \) is in \( g \) for some \( g \in G \). In this case we say that \( G \) is an open cover of \( [0,1] \). Show there is a finite sub-collection of \( G \) that also covers \( [0,1] \). A set that has the property that each open cover has a finite subcover is called \textit{compact}.

a) Replace open interval with open set.

b) A number set is compact iff it is closed and bounded.

41. The sequence \( S \) has limit \( p \) iff each subsequence \( T \) has limit \( p \).

42. If the sequence \( S \) is bounded, then \( S \) has a subsequence \( T \) which has a limit.

43. If \( T \) is a sequence with limit \( t \) and \( R \) is a sequence with limit \( r \), then the sequence \( S \) defined by \( S(n) = T(n) + R(n) \) has limit \( t + r \).

\textbf{Def.} If \( D \) is a subset of \( \mathbb{R} \) and \( f \) is a real valued function with domain \( D \) then the statement that \( f \) \textit{is continuous at the point} \( p \) in \( D \) means if \( (a,b) \) contains \( f(p) \), then there is a \( (c,d) \) containing \( p \) such that \( f(x) \) is in \( (a,b) \) for each \( x \) in \( D \cap (c,d) \).

\textbf{Def.} The statement that \( f \textit{is continuous} \) means that \( f \) is continuous at each point in its domain.

\textbf{Def.} If \( D \) is a subset of \( \mathbb{R} \), then the statement that \( U \) is open/closed relative to \( D \) means there is an open/closed set \( V \) such that \( U = V \cap D \).

44. Suppose \( D \) is a subset of \( \mathbb{R} \), \( f \) is a real valued function with domain \( D \), and \( p \) is in \( D \). The following statements are equivalent:

a. \( f \) is continuous.

b. if \( p \) is in \( D \) and \( c > 0 \) then there is a \( d > 0 \) such that if \( x \) is in \( D \) and \( |x - p| < d \) then \( |f(x) - f(p)| < c \).

c. if \( U \) is an open set, then \( f^{-1}(U) \) is open relative to \( D \).

d. if \( K \) is a closed set, then \( f^{-1}(K) \) is closed relative to \( D \).

e. if \( p \) is in \( D \) and \( S \) is a sequence in \( D \) with limit \( p \), then the sequence \( T \) defined by
\[ T(n) = f(S(n)) \] has limit \( f(p) \).

45. The sum of continuous functions is continuous over a common domain.
46. The product of continuous functions is continuous over a common domain.
47. The composition of continuous functions is continuous provided the domain for the outside function contains the image of the inside function.
48. There is a function defined on \( \mathbb{R} \) that is continuous at a single point only.
49. There is a non-decreasing function \( f \) defined on \([0,1]\) such that \( f \) is continuous at infinitely many points and discontinuous at infinitely many points.
50. If \( f \) is continuous over \( U \) and \( V \) is contained in \( U \), then \( f \) restricted to \( V \) is continuous.
51. If \( f \) is continuous over \( D \) then there is an \( x \) in \( D \) such that \( f(x) > f(y) \) for each \( y \) in \( D \).
   
   Showed to be false.
51 a. If \( f \) is continuous over \( D \) and \( D \) is bounded then same conclusion. Shown to be false as well.
51b. If \( f \) is continuous over the compact set \( D \), then the image of \( f \) is compact.
52. If \( f \) is continuous over \( D \) and \( A \) is a closed subset of \( D \) then \( f(A) \) is closed. Shown to be false
52a. replace “closed” with “compact”.
53. If \( f \) is continuous over \( D \) and \( f(a) < y < f(b) \), then there is a \( c \) in \( D \) such that \( y = f(c) \).
   
   Shown to be false
53a. Replace arbitrary \( D \) with a closed interval.
54. If \( f \) is continuous over the compact set \( A \) and \( c > 0 \), then there is a \( d > 0 \) such that if each of \( x \) and \( y \) is in \( A \) and \( |x - y| < d \), the \(|f(x) - f(y)| < c\).
55. There is a function \( f \), defined over \([0,1]\) such that \( f \) is continuous on the irrational numbers and discontinuous on the rationals.
56. If $f : [0,1] \rightarrow \mathbb{R}$ is non-decreasing and not continuous, then \{x in [0,1]: f is not continuous at x\} is countable.

57. There is an $f : [0,1] \rightarrow \mathbb{R}$ such that f is continuous and f “cannot be drawn”. **Test 2 # 5 iv.**

Math 432
Open problems from 431:
T7. Every “random” T-set is homeomorphic to the T-set.

42. If the sequence S is bounded, then S has a subsequence T which has a limit.

51b. The continuous image of a compact set is compact.

53a. If f is continuous over [a,b] and f(a) < y < f(b), then there is a c in D such that y = f(c).

54. If f is continuous over the compact set A and c > 0, then there is a d > 0 such that if each of x and y is in A and |x - y| < d, the |f(x) - f(y)| < c.

55. a. There is a non-decreasing function f, defined over [0,1] such that f is continuous on the irrational numbers and discontinuous on the rationals.

56. If $f : [0,1] \rightarrow \mathbb{R}$ is non-decreasing and not continuous, then \{x in [0,1]: f is not continuous at x\} is countable.

**New Problems**

**Def.** The statement that the set A is homeomorphic to the set B means there is a continuous bijection from A onto B (and vice-versa).

58. (0,1) is homeomorphic to R. [0,1] is not.

59. There is a continuous function defined on [0,1] which is neither increasing nor decreasing (nor constant) over each subinterval of [0,1].

60. If A is compact and $f : A \rightarrow B$ is a continuous bijection, then $f^{-1}$ is continuous.

   a. Why can’t we just use closed instead of compact?

61. If A is a dense subset of the [0,1] and $f : A \rightarrow \mathbb{R}$ is continuous, then there is a continuous $g : [0,1] \rightarrow \mathbb{R}$ such that $g$ restricted to A is $f$.

**DEF.** Suppose $f : [a,b] \rightarrow \mathbb{R}$ and $x \in (a,b)$. The statement that $f$ has slope at $x$ means there is a number m such that the following function is continuous at $x$. 
\[ g(y) = \begin{cases} 
\frac{f(x) - f(y)}{x - y}, & y \in [a, b] \\
\frac{m}{y}, & y = x 
\end{cases} \]

62. If each of \( f \) and \( g \) has slope at \( p \) then:
   a) \( f + g \) has slope at \( p \).
   b) \( fg \) has slope at \( p \).

63. If \( g \) has slope at \( p \) and \( f \) has slope at \( g(p) \) then \( f \circ g \) has slope at \( p \).

64. Is there a function defined on \([0,1]\) with slope at exactly one point?

65. If \( f \) has slope at \( p \), then \( f \) is continuous at \( p \).

66. If \( f: [0,1] \to \mathbb{R} \) and \( f \) has a max at \( c \) then either \( f \) has no slope at \( c \) or the slope of \( f \) at \( c \) is 0.

67. If \( f \) has slope on all of \([0,1]\) and \( f(0)=f(1) \), then there is an \( x \) in \((0,1)\) such that the slope of \( f \) at \( x \) is 0.

68. If \( f \) has slope on \([a,b]\) then there is a \( c \) in \((a,b)\) such that
   \[ f'(c)(b-a) = f(b) - f(a), \text{equivalently, } f'(c) = \frac{f(b) - f(a)}{b-a}. \]

**Def** Let \( D \) be a subset of the reals and for each positive integer \( i \), let \( f_i: D \to \mathbb{R} \). The statement that this sequence of functions **converges pointwise** means that for each \( p \) in \( D \) the number sequence \( S_p \) defined by \( S_p(n) = f_i(p) \) has limit.

69. There is a sequence of continuous functions that converge pointwise to a non-continuous function.

**Def** The statement that \( S \) is a metric space means there is a function \( d: S \times S \to \mathbb{R} \) such that
   i. \( d(s,s) = 0 \) for each \( s \) in \( S \)
   ii. \( d(s,r) \geq 0 \) for each \( r \) and \( s \) in \( S \)
   iii. \( d(r,t) \leq d(r,s) + d(s,t) \) for each \( r, s, \) and \( t \) in \( S \)

Note: The usual definition of distance between points in \( \mathbb{R}^2 \) and \( \mathbb{R}^n \) form a metric for those spaces. That is, the topology of the line can easily be extended to Euclidean 2-space (ie. \( \mathbb{R}^2 \) or any other metric space for that matter, using the following definition of a basic open set).

**Def** If \( p \) is a point and epsilon is greater than zero, then the epsilon ball centered at \( p \) is defined to be the set of points within epsilon of \( p \). This is also referred to as the epsilon neighborhood of \( p \).
This definition naturally extends the topology of the line to that of the plane, and beyond. Of course, since we also have limit points defined in terms of distance, as long as we have the concept of distance, which we already do in Euclidean n-space, it is possible to mimic many of the theorems from 431 to higher dimensional spaces.

Also, the “positive number c definition” of limit convergence can be extended to each metric space. Axiom 1 from 431 essentially states that the real numbers do not contain a “hole”. A more instructive way of viewing this statement might be to say that each Cauchy Sequence in R converges to a point in R (problem 39).

70. Prove the following problems from 431 for Euclidean 2-space.
   a) If A is infinite and bounded, then A has a limit point (Problem 4b)
   b) \( \mathbb{R}^2 \) is complete. (see def below. Problem 39)
   c) The closed ball of radius 1 is compact. (Problem 40)
   d) The intersection of a monotonically non-increasing sequence of compact sets has a point in common (Problem 28b.)

**Def** The statement that the metric space S is complete means that each Cauchy Sequence in S converges to a point in S.

The results of problem 69 lead us to believe that if we wish to build a complete space of continuous functions over \([0,1]\), we need to build a suitable metric, one that goes beyond pointwise convergence for example. In order to do this, think about what it might mean for 2 functions in \(C[0,1]\) to be within epsilon of each other.

71. Define a metric on \(C[0,1]\) such that \(C[0,1]\) represents a Complete metric space.

72. There is an \(f:[0,1] \rightarrow \mathbb{R}\) such that \(f\) has slope at each \(x\) in \([0,1]\) but such that the derivative of \(f\) is unbounded.

73. \(C[0,1]\) is separable.

74. Is there an \(f:[0,1] \rightarrow \mathbb{R}\) such that \(f\) has slope at each \(x\) in \([0,1]\) but its derivative is not continuous on \([0,1]\)?

**Def** If \([a,b]\) is a closed interval, then by a *subdivision of \([a,b]\)* we mean a finite collection D of non-overlapping closed intervals whose union is \([a,b]\). (Two closed intervals are nonoverlapping provided either they do not intersect or their intersection is a single point.) An equivalent definition is that a subdivision of \([a,b]\) is a finite increasing sequence in \([a,b]\) such that its initial point is a and is final point is b.

**Def** The statement that the subdivision D refines the subdivision D′ means that if d is in D then there is an e in D′ such that d is a subset of e. In this case we say that D is a refinement of D′. (Equivalently using the sequence definition, D refines D′ means that D′ is a subsequence of D.)

**Def** If D is a collection of sets, then a *choice function* for D is a function ch from D into the union
of $D$ with the property that $\text{ch}(d)$ is in $d$ for each $d$ in $D$.

**Def** If $D$ is a subdivision of $[a,b]$, then the *mesh* of $D$ is $\max\{|d|:d \text{ is in } D\}$. ($|d|$ denotes the length of $d$.)

**Def** Suppose $[r,s]$ is a closed interval and $g$ is a real valued function whose domain contains $[r,s]$. the *g-length* of $[r,s]$ is the number $g(r)-g(s)$ and will be denoted by $g[r,s]$.

We are now in a position to define two (possibly different) notions of integral. Suppose $I=[a,b]$ and each of $f$ and $g$ is a real valued function defined on $I$.

**Def** The statement that $f$ is integrable on $I$ with respect to $g$ means there is a number $m$ such that if $\varepsilon > 0$ there is a $\delta > 0$ such that if $D$ is a subdivision of $I$ with mesh less than $\delta$ and $\text{ch}$ is a choice function for $D$, then $\left| \sum_{d \in D} f(\text{ch}(d)) \ g \ |d - m| \right| < \varepsilon$. In this case we will denote the number $m$ by $\int_a^b f \ dg$.

**Def** The statement that $f$ is type-R integrable on $I$ with respect to $g$ means there is a number $m$ such that if $\varepsilon > 0$ there is a subdivision $D$ of $I$ such that if $D$ refines $D$ and $\text{ch}$ is a choice function for $D$, then $\left| \sum_{d \in D} f(\text{ch}(d)) \ g \ |d - m| \right| < \varepsilon$. In this case we will denote the number $m$ by $\int_a^b f \ dg$.

75. a. If $f$ is integrable on $I$ with respect to $g$ then the number $m$ is unique. Aja

b. Are the two types of integrals the same? In particular, if $f$ is both integrable and type-R integrable, must the integrals be equal? Are there $f$, $g$, and $I$ such that $f$ is integrable with respect to $g$ on $I$ but not type-R integrable, or vice-versa?

c. $\int_a^b cf \ dg = c \int_a^b f \ dg$.

d. $\int_a^b (f + g) \ dh = \int_a^b f \ dh + \int_a^b g \ dh$.

e. $\int_a^b fd (g + h) = \int_a^b f \ dg + \int_a^b f \ dh$.

f. $\int_a^b f \ dcg = c \int_a^b f \ dg$.

76. Find a function $f:[0,1] \to \mathbb{R}$ such that $f$ is not integrable.
77. If \( f : [0.1] \rightarrow \text{non-negative reals} \) is continuous, \( g \) is non-decreasing, \( f(\frac{1}{2}) > 0 \), and \( \int_0^1 f \, dx \) exists, then \( \int_0^1 f \, dx > 0 \). Is this true if we replace \( dx \) with \( dg \) for each non-decreasing \( g \)?

78. If \( \int_a^b f \, dg \) exists then \( f \) is continuous.

79. Is there and \( f : [0,1] \rightarrow \mathbb{R} \) such that \( f \) is bounded, \( f^2 \) is Riemann integrable, but \( f \) is not Riemann integrable?

**Def** The statement that the function \( f : [a,b] \rightarrow \mathbb{R} \) is of **bounded variation** (denoted by B.V.) means that there is a number \( M \geq 0 \) such that if \( x_1, x_2, \ldots, x_n \) is a subdivision of \([a,b]\) then
\[
\sum_{i=1}^{n-1} \left| f(x_{i+1}) - f(x_i) \right| < M.
\]

80. There is a continuous function defined on \([0,1]\) which is not B.V.

81. If \( f : [0,1] \rightarrow \mathbb{R} \) is non-decreasing, then \( f \) is B.V.

82. If \( f : [0,1] \rightarrow \mathbb{R} \) and \( a \in (0,1) \) and each of \( f_{[0,a]} \) and \( f_{[a,1]} \) is B.V. then \( f \) is B.V.

83. If \( f \) is B.V. on \([0,1]\) then there are non-decreasing functions \( h \) and \( g \), each defined on \([0,1]\), such that \( f = h - g \).

84. If \( f : [0,1] \rightarrow \mathbb{R} \) is B.V. and \( x \) is in \([0,1]\), then \( f_{[0,x]} \) is B.V.

85. a. \( \int_0^1 f \, dx \) exists for each continuous \( f \).

b. \( \int_0^1 f \, dg \) exists for each continuous \( f \) and non-decreasing \( g \).

c. \( \int_0^1 f \, dg \) exists for each continuous \( f \) and BV\( g \).

86. Show \( f(x) = \begin{cases} 0 & , 0 \leq x < .5 \\ 1 & , .5 \leq x \leq 1 \end{cases} \) is Riemann integrable over \([0,1]\).

87. There is an \( f : [0,1] \rightarrow \mathbb{R} \) such that \( f \) has infinitely many points of discontinuity and is Riemann integrable.

a. There is such an \( f \) that is non-decreasing.

b. There is such an \( f \) that is discontinuous at each rational.