Math 311 Course Outline
UHH Mathematics Department

Course Description


Prerequisites

Math 206 or consent of instructor. (Math 310 or CS 215 is also desirable.)

Course Goals

The primary goal is to introduce the students to the rigorous theorem-based structure of advanced mathematics. The secondary goal is for students to become familiar enough with the particular structure of linear spaces so as to be able to recognize non-standard linear spaces such as $C[0,1]$ or the solution space for ordinary homogenous linear differential equations, and to use general linear algebra knowledge gained in this course to investigate these spaces.

Use of Technology

Given the primary goals of the course above, the role of technology should be to simplify numerical calculations (e.g. computing determinants, matrix inverses, or Eigenvalues/Eigenvectors) for either student work or instructional purposes.

Primary Topics

Systems of Linear Equations
Determinants
Vector Spaces
Linear Transformations
Eigenvalues and Eigenvectors

The latter three topics are by far the most important; but they require the ability to solve linear systems, and an understanding of the implications of the determinant to such solutions. Given time constraints, therefore, every effort should be made to quickly proceed through the first two areas of study.

Optional Topics

Graham-Schmidt Orthogonalization
Inner Products
Course Objectives

Systems of Linear Equations (2 weeks)

The primary objective of this section is to provide the student with enough background to be able to determine linear dependency and to compute the rank and nullity of Linear Transformations later in the course. The work is comprised of two distinct but complimentary areas, the mechanics of using matrices to solve linear systems and gaining an introductory understanding of the structure of such solutions. The areas can be represented as follows:

- Matrix representation of linear systems
- Basic row reduction
  - The mechanics of Gauss-Jordan row reduction
  - How to represent infinite solutions
- The use of matrix inverses to solve systems with unique solutions
- Theorems regarding the structure of the solution space with an eye towards later sections regarding rank and kernel of linear transformations, including but not limited to:
  - The main structure theorem…the solutions to non-homogeneous systems are of the form A + B where A is a single solution to the non-homogeneous system and B is the [possibly trivial] space of solutions to the associated homogeneous system.

Determinants (2 weeks)

The primary objectives of this section are to learn to compute determinants, the algebra of determinants (e.g. \( \text{Det}(AB) = \text{Det}(A) \text{Det}(B) \)), and the implications of the determinant to solutions of systems of linear equations (i.e. non-zero determinant implies unique solution, zero determinant implies either no solution or infinite solutions). Ample time generally does not exist to cover all the many properties of determinants, or even Cramer’s rule, which is a favorite of Engineers, but not necessarily mathematicians. Technology can be very useful when computing determinants, either in class or on assigned homework.

- Co-factor expansion (since it generalizes past the 3x3)
- The relationship between the determinant and the existence of inverses, and the structure of the solution to linear systems

Note: The determinant is needed later to formulate the characteristic equation in order to compute Eigenvalues.

Vector Spaces (5 weeks)

This chapter represents the students’ first attempt at independent proof. \( \mathbb{R}^n \) should be introduced as a concrete example, but other algebraic definitions for ‘plus’ and ‘scalar
multiplication’ should be emphasized. Matrices form an elementary starting point for this, but more theoretical spaces such as function spaces should be emphasized. Care should be taken to identify the ‘large’ spaces as vector spaces, such as the function spaces from \( \mathbb{R}^m \) to \( \mathbb{R}^n \) under the usual function operations, or the space of \( n \times m \) matrices under the usual operation. This allows the amount of work the students need to show when demonstrating, for example, that the set of all \( 3 \times 3 \) diagonal matrices form a vector space, or the set of real valued differentiable functions form a vector space. Once the concept of vector space/subspace is under control, attention should shift to bases, which requires a good understanding of linear independence/dependence.

- Determining whether an ordered Triple \( (S, \oplus, \otimes) \) is a vector space
  - Use a variety of examples, both standard and non-standard (particularly with respect to the operations)
- Subspace theorem
  - Emphasize the need to only show closure of the two operations
  - Ensure students know the difference between subspace and subset
- Linear Combinations
- Span
  - Associated Theorems (e.g. equivalent cardinality of two linear independent spanning sets, what if a set spans a space – must it be linearly independent, etc.)
- Linear dependency and bases
  - Use matrices and determinants to determine if sets of vectors are linearly independent
  - Identify canonical bases for non-standard spaces such as \( \mathbb{P}_2 \).

**Linear Transformations (4 weeks)**

The purpose of this chapter is to gain an understanding of linear transformations as linear functions operating on spaces such as \( \mathbb{R}^n \). Ultimately, the goal is to use eigenvectors and Eigenvalues to help in identifying the effects on an object operated on by such a function. Thus, this chapter should be spent familiarizing the student with how to identify when a function is linear or not, the implications to computing values once it is determined that the transformation is linear (i.e. the image of a basis determines the entire image), and the use of the Ker and Rank to identify the dimension of the image of a transformation. Examples should include important operators such as the derivative.

- Identify whether or not \( T \) is linear.
- Understand the implications of \( T \) being linear.
  - Compute \( T(X) \), given \( T(B) \), where \( B \) is a [not necessarily canonical] basis for \( \text{Dom}(T) \).
  - Describe geometrically \( T(A) \), where the boundary of \( A \) is piecewise linear in \( \mathbb{R}^2 \).
- Compute \( \text{Ker}(T) \), Nullity(\( T \)), and Rank(\( T \))
  - Note: in many instances this involves solving homogeneous linear systems, again back to section one.
- Describe geometrically Ker(T)
- Compute matrix representation for T

**Eigenvalues and Eigenvectors (3 weeks)**

The purpose of this chapter is to extend the work of the previous chapter to better understand the behavior of linear transformations, in particular with respect to identifying the image of (T).

- Compute Eigenpairs for 2 x 2 and 3 x 3 matrices.
- Find bases for Dom(T) using eigenvectors
- Geometrically describe Im(T) using eigenpairs
- Find bases for Im(T) using eigenvectors
- Implications of zero eigenvalue to Im(T)
- Diagonalize 2x2 and 3x3 matrices using eigenpairs
  - Diagonalize A, and compute A^n (optional)
  - Symmetric Matrices
  - Quadric Forms (Optional)