Program Review 2012
Mathematics Department Assessment Efforts

Introduction

The Mathematics Department has engaged in multiple assessment efforts since the last program review, including efforts in its major program and its General Education/Service mission. As part of the 2008 campus-wide assessment effort the Department developed student learning outcomes for its major program for both the traditional and teaching tracks, and developed a curriculum matrix that aligned those outcomes with its courses. At the same time the department developed a schedule for assessing the various outcomes throughout subsequent years, usually choosing two – three outcomes per year. Unfortunately, the department did not follow the initial assessment schedule. Recent developments have also required departments to assess institutional General Education goals. Fortunately, there is overlap in these two types of assessments. Our Calculus classes, for example, are an integral part of our major program, but are also part of the GenEd program and service numerous NS departments and Pharmacy. Thus, in many instances assessments for the program also gave us information regarding how well we are meeting institutional goals. The department engaged in eight assessment activities in recent years, four of which spanned more than one semester and one that spanned six semesters.

What follows is a detailed report of all assessment efforts since the last program review. Additionally, it is of particular interest to note that the department is striving to make assessment a more intuitive part of our work, involve a higher percentage of the faculty, and encourage more discussion of how well we are meeting our mission. To that end the department has adopted a user-friendly annual assessment report form that is used to identify two assessments each year and the faculty performing the assessments, and makes the reporting process and resulting discussions easier, and it has created a more recent assessment plan with complete department participation. It is the hope that such a form and plan will not only encourage broader participation, spreading the assessment load, but will help to departmentalize assessment as a natural part of what we do.

Glossary of abbreviations:

SLO – Student Learning Outcomes (in general)
CLO – Course Learning Outcomes (SLO’s from specific courses)
ILO – Institutional Learning Outcomes (Appendix A-Institutional Learning Outcomes)
PLO – [Math Department] Program Learning Outcomes (Appendix A – Curriculum Matrix)

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1 See Appendix A - Curriculum Matrix.
2 See Appendix A - Math Department Major Assessment Timeline.
3 See Appendix A - Annual Assessment Report Form.
Summary of assessment efforts

All of the assessments except the one related to Analysis (math 431-432) were direct assessments that embedded problems that address specific learning outcomes or institutional goals directly into course exams. At least two faculty members met in each case to discuss the types of problems that should be assessed and the associated learning outcomes, and rubrics were jointly determined prior to scoring. In most instances, however, very little whole-department discussion occurred as a result of the assessments. This is a reflection of the department’s indifferent “state of assessment readiness”, which is currently only at an emerging level if viewed from the perspective of developing a “culture of assessment” as desired by WASC. Consequently, although the assessments were perhaps valuable to the faculty involved, who then used the results to “close the loop” and make adjustments to their teaching, the department is still struggling implementing methods that would close the loop on a broader basis.

Each of the assessment efforts provided valuable information about how the department is reaching its learning outcomes goals, particularly within certain courses. None of the assessments identified critical weaknesses, which given our record of teaching excellence is not entirely unexpected. The most useful assessments identified weaknesses in student understanding, usually in terms of the Analysis and Critical Thinking Institutional Goals and sometimes with respect to Program Learning Outcomes. The most detailed information of this type was gained in Math 206 and Math 311, Calculus II and Linear Algebra, respectively, which the department considers critical courses in terms of both the major and in serving Natural Sciences. The instructor(s) closed the loop in these two courses by addressing these weaknesses. The scores over multiple semesters showed marked improvement in some cases.
Compilation of Assessments since the last Program Review

<table>
<thead>
<tr>
<th>Timeframe</th>
<th>Faculty Involved</th>
<th>Classes Involved</th>
<th>SLO’s Assessed and Assessment Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>F 2011</td>
<td>Brian Wissman, Shuguang Li, Zorana Lazarevic, Mitchell Anderson</td>
<td>Math 205</td>
<td>QR ILO’s (Computation) Embedded Assessment</td>
</tr>
<tr>
<td>F 2008 and F 2011</td>
<td>Efren Ruiz, Mitchell Anderson</td>
<td>Math 205</td>
<td>QR ILO’s (Computation) Gateway Derivative Exams</td>
</tr>
<tr>
<td>F 09, Sp 10 (2), Su 11, F 11, Sp 12, Su 12, Sp 12</td>
<td>Mitchell Anderson, Brian Wissman</td>
<td>Math 206</td>
<td>QR ILO’s (all 3 and Critical Thinking) Embedded Assessment</td>
</tr>
<tr>
<td>Sp 2012</td>
<td>Brian Wissman, Mitchell Anderson, Shuguang Li</td>
<td>Math 206</td>
<td>QR ILO’s (all 3 &amp; Critical Thinking), Comm ILO’s (Organization &amp; Structure, Line of Reasoning) Group Project</td>
</tr>
<tr>
<td>Sp 2012</td>
<td>Brian Wissman, Mitchell Anderson</td>
<td>Math 206</td>
<td>QR ILO’s (all 3 and Critical Thinking) Embedded Assessment</td>
</tr>
<tr>
<td>Sp 2010, Sp 2012</td>
<td>Mitchell Anderson, Brian Wissman</td>
<td>Math 311</td>
<td>QR ILO’s (all 3 and Critical Thinking) PLO’s – All Embedded Assessment</td>
</tr>
<tr>
<td>F 2010 – Sp 2011</td>
<td>Mitchell Anderson, Brian Wissman, Roberto Pelayo, Efren Ruiz</td>
<td>Math 431-432</td>
<td>QR ILO’s (Analysis and Critical Thinking) PLO’s – All Course Portfolio</td>
</tr>
</tbody>
</table>
The details of each assessment are provided using a common template developed specifically for this purpose.

**Math 104 Assessment**

1. Timeframe for the assessment(s).
   - Spring 2011, Spring 2012.

2. Faculty involved.
   - Diana Webb and Zorana Lazarevic administered the assessment for Spring 2011, Diana Webb and Efren Ruiz administered the assessment for Spring 2012, and Mitchell Anderson provided oversight and assistance for both.

3. Student Learning Outcomes Assessed
   - All Quantitative Reasoning ILO’s, including critical thinking.
   - The following CLO’s
     - Apply function notation, particularly for the composition of functions
     - Recognize the standard functions and be able to recognize what sets each apart from others (polynomials for this assessment)
     - Solve [exponential and] logarithmic equations
     - Solve triangles
     - Solve trigonometric equations

4. Courses in which the assessment was administered.
   - Math 104F, Math 104G, and Math 104

5. Audience (i.e. Math majors, NS majors, non-science majors), levels (i.e., freshmen/soph or junior/senior), and number of participants.
   - Students in these courses are typically freshmen or sophomore NS majors.
   - For Spring 2010, 70 students from three sections and two instructors took the 104F portion of the assessment, and 51 students from two sections and a single instructor took the 104G portion of the assessment. 41 students took the Math 104F final exam, 22 students took the Math 104G final exam, and 29 students took the Math 104 final exam, which included both the Math 104F and Math 104G problems.

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4 See Appendix A – Math 104 Course Outline
For Spring 2011, 64 students from three sections and two instructors took the 104F portion of the assessment, and 28 students from one section and a single instructor took the 104G portion of the assessment. Of the 64 students taking the Math 104F portion, 36 were enrolled in Math 104F and the remaining 28 were enrolled in Math 104. Those in Math 104 were the only students who took part in the Math 104G assessment.

6. Details of the Assessment

➢ What type of assessment was administered (Direct or Indirect), and how was the data collected?

- A direct assessment was employed for each semester by embedding common problems into the final exams for multiple sections of Math 104, 104F, and 104G. The same problems were used each year. Three problems dealing with functions were embedded into the Math 104F finals and two trigonometry problems were embedded into the Math 104G finals. Students in Math 104 received all five problems.

➢ How was the assessment developed?

- Diana and Zorana met to identify three problems from Math 104F material to be embedded in the Math 104F/104 final exams, and two problems from Math 104G to be embedded in the Math 104G/104 final exams. The 104F problems assessed students’ abilities to use function notation and composition, the behavior of polynomials represented in factored form with repeated roots included, and using the properties of the log function to solve log equations. The 104G problems assessed solving right triangles and solving a trig equation with a multiple of the variable x.

- The problems were embedded on the last page of the final exams to make their appearance uniform across instructors and sections, to better facilitate copying after the exams were turned in, and to help assure student anonymity (i.e. no names were present on the last page).

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5 The Spring 2010 assessment was undertaken at the request of the UHH Assessment Support Committee and was intended as a pilot to test the potential for using the newly developed Quantitative Reasoning and Scientific Inquiry institutional goals (Appendix A – Institutional Learning Objectives for Quantitative Reasoning).

6 Appendix -Math 104F Assessment Problems and Rubrics, and Math 104G Assessment Problems and Rubrics
The same problems were used for the second semester, Spring 2011.

How was it analyzed? (e.g. What type of scoring rubric was developed, who developed it, and who did the scoring?)

- Diana, Zorana, and Mitchell developed scoring rubrics for each problem by assigning points to various concepts and steps associated with each problem, with each test question receiving integer scores of between 0 and 4 for four of the problems and between 0 and 3 for the remaining problem. They then aligned each problem to one or more of the Quantitative Reasoning ILO’s (Calculations, Analysis, and Visual); each problem also corresponded to a course CLO (provided above). The numerical average percentage scores were then translated to achieving various levels for the ILO’s and CLO’s as designated in the following table:

<table>
<thead>
<tr>
<th>Raw Score</th>
<th>ILO or CLO equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 – 25%</td>
<td>Beginning</td>
</tr>
<tr>
<td>26% - 50%</td>
<td>Emerging</td>
</tr>
<tr>
<td>51% - 75%</td>
<td>Competent</td>
</tr>
<tr>
<td>76% - 100%</td>
<td>Advanced</td>
</tr>
</tbody>
</table>

- For each semester, Mitchell joined the faculty members to individually score each problem according to the rubrics. They then compared their results for each student and each problem, reconciling any differences. Averages of the totals were compiled. The same scoring rubric was used each semester.

7. Results and analysis.

One important result for this assessment was validation of the Quantitative Reasoning Institutional Goal, including the Critical Thinking component, as developed by the UHH Faculty Congress’ Assessment Support Committee. The Assessment Committee was grateful to have a working model that utilized the new institutional goals (ILO’s) and rubrics, and confidently submitted their work to the UHH Congress for approval. Mitchell sent a final report on the first semester pilot project to Dr. Seri Luangphinhith, Chair of the Assessment Support Committee and it was later included in a WASC

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7 Critical Thinking, another Institutional Goal that crosses over most other institutional goals, is also recognized as being satisfied by the Analysis and Visual components of Quantitative Reasoning and Scientific Inquiry. It is not treated separately within our rubric.
Resource Binder for use in their Assessment 101 workshop held at the Waikiki Beach Marriott in Honolulu, Feb. 2012.  

Results for the Spring 2010 104 Assessment

<table>
<thead>
<tr>
<th></th>
<th>Math 104F</th>
<th></th>
<th>Math 104G</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>70 Exams</td>
<td>51 Exams</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Possible Score</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>Average</td>
<td>2.71</td>
<td>3.38</td>
<td>2.81</td>
<td>2.39</td>
</tr>
<tr>
<td>ILO’s Assessed</td>
<td>Calculations</td>
<td>Analysis and Visual</td>
<td>Calculations</td>
<td>Calculations and Visual</td>
</tr>
<tr>
<td>CLO’s Assessed</td>
<td>Function Composition</td>
<td>Properties of Polynomials</td>
<td>Log Equations</td>
<td>Solve Triangles</td>
</tr>
<tr>
<td>Quantitative Reasoning Translation</td>
<td>Competent</td>
<td>Advanced</td>
<td>Competent</td>
<td>Advanced</td>
</tr>
</tbody>
</table>

It was interesting and perhaps gratifying to note that the students performed better on the conceptual material that required the use of visual critical thinking, regardless of whether or not the problem came from 104F material or from 104G. Students performed satisfactorily on all problems except G2, reaching or approaching competency. A discussion of G2 is provided below under closing the loop.

It should be noted that these are aggregate scores and that as expected the standard deviation for this particular data grew for lower scores, indicating that many students in that instance scored much higher than the mean. However, no deep statistical analysis was performed for the data.

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This second assessment compares favorably in some aspects to the first, and less so in other aspects. This variation is somewhat expected given the difference in instructors and their differing emphases, and of course student aptitude is also not uniform across classes. The results for function composition, a purely symbolic concept in this assessment, improved from 2.71 to 2.89, but the results for polynomials and log equations dropped significantly, almost 50% and 40% respectively. The Trigonometry results remained almost identical, which again is not surprising since the same instructor taught Trig for both studies. The results indicate a possible weakness in these two areas, properties of polynomial functions and solving logarithmic equations. See the discussion below on this matter in Closing the Loop.

8. Closing the loop.

The three faculty members involved in the Spring 2010 assessment discussed the results of that pilot. They acknowledged that the students performed satisfactorily in all but one area. With respect to problem G2, in which students were asked to solve a trigonometric equation involving a double angle (one of the last topics covered in Math 104 and one of the most difficult), the instructor noted that s/he intended to try to allot more time for that subject in the future.

The results on the first problem in 104F and both in 104G were very much the same the second time as the first. The trigonometry instructor did mention that time constraints did not allow the extra time hoped for after the first assessment to spend more on the final problem; with [not surprisingly] virtually identical scores as a result. The biggest surprise was the lower scores for the conceptual polynomial problem, number two in the
104F assessment. Much of this can be attributed to the fact that one instructor did not present the polynomial material from this conceptual orientation. However, later deliberate discussions with department members on this very topic resulted in some good ideas on teaching this subject in the future. In other words, the instructor has identified a weakness and perfectly understands how to address it the next time around.

**Math 205 Assessment #1 of 2**

1. Timeframe for the assessment(s).
   - Fall 2011

2. Faculty involved.
   - Shuguang Li, Brian Wissman, Zorana Lazarevic, and Mitchell Anderson.

3. Student Learning Outcomes Assessed
   - ILO – Computational and Analysis
   - The following CLO’s
     - Demonstrate proficiency in using the standard differentiation rules
     - Utilize u-substitution when appropriate

4. Courses in which the assessment was administered.
   - Math 205

5. Audience (i.e. Math majors, NS majors, non-science majors), levels (i.e., freshmen/soph or junior/senior), and number of participants.
   - Students in these courses are typically freshmen or sophomore NS majors.
   - 124 students from six sections and four instructors participated in this assessment.

6. Details of the Assessment
   - What type of assessment was administered (Direct or Indirect), and how was the data collected?

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9 See Appendix – Math 205 – 206 Course Outline
A direct assessment was employed by embedding three common problems into the final exams. This was not a major assessment effort but was intended to provide some experience to department members with the assessment processes.

How was the assessment developed?

After discussions between the four instructors Brian identified the problems and grading rubrics and presented them to Shuguang, Zorana, and Mitchell, who agreed they were at the appropriate level. The first two problems were product and quotient rule differentiation problems, respectively, and the third problem was a u-substitution anti-differentiation problem. All three problems aligned primarily with the Calculations part of the Quantitative Reasoning ILO’s, but also involved the Analysis portion as well since students cannot apply differentiation rules without understanding the role played by each function and algebraic operation, and cannot correctly apply u-substitution without understanding the chain rule.

The intent was to have the three problems embedded into the final exams for all sections. Unfortunately, poor communication and time constraints resulted in one instructor assigning the problems online via MyMathLab. For those sections about 25% of the students did not participate in the optional assessment, which may have slightly skewed the results, and in such cases questions always arise regarding the improper use of notes or other inappropriate resources. Nevertheless, the results from those sections were not inconsistent with the others, and rather than report them separately or throw them out entirely, the team decided to include them and to suggest that in the future all embedded assessments attempt to be as uniform as possible.

Note: Dr. Anderson and Dr. Wissman met in 2009 to develop embedded assessment problems for future use by the department in assessing student progress in Math 205. However, the problems they developed were intended for a more comprehensive (i.e. throughout the semester) assessment that concentrated more on key concepts rather than manipulation skills. It is suggested that these assessment problems be used by the department for a future, more comprehensive, Math 205 assessment effort.

How was it analyzed? (e.g. What type of scoring rubric was developed, who developed it, and who did the scoring?)

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10 See Appendix – Math 205 Embedded Assessment Problems and Rubrics.
11 See Appendix – Math 205 Suggested Comprehensive Assessment Problems.
Brian developed the scoring rubric, which was accepted by Shuguang, Zorana, and Mitchell. Since the scoring rubric was very straightforward, and the end of the semester was at hand, each instructor graded their own students’ work based on the rubric, and then passed the results to Brian for compilation. It was agreed that a score of 0 or 1 translated as Beginning level, 2 or 3 as Emerging, and 4 or 5 as Competent. The level of these problems did not warrant an Advanced rating for any of the three problems.

7. Results and analysis.

Results for Fall 2011 205 Assessment

<table>
<thead>
<tr>
<th></th>
<th>Math 205</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Product Rule</td>
<td>Quotient Rule</td>
</tr>
<tr>
<td>Possible Score</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>Average</td>
<td>4.10</td>
<td>3.34</td>
<td>2.72</td>
</tr>
<tr>
<td>QR Translation</td>
<td>Competent</td>
<td>Approaching Competent</td>
<td>Emerging</td>
</tr>
</tbody>
</table>

The average scores for the differentiation problems indicated students either reached or approached competency, which is expected for these types of Math 205 problems. Approximately 76% of the students scored a 4 or 5 for the product rule, and 52% for the quotient rule. The product rule results were consistent with instructor expectations, but quotient rule results were lower than expected. Approximately 46% of the students earned a score of 4 or 5 for u-substitution. Given that u-substitution is the most difficult of the three and is the very last subject covered in 205, the results for u-substitution are consistent with expectations. It is important to note that u-substitution is again covered in the subsequent course, Math 206, and that students there generally consider it to be one of their easiest techniques to apply.

8. Closing the loop.

The results of the assessment were presented to the department for discussion in Fall 2012. Not surprisingly, they did not generate much discussion on strengths and weaknesses. However, it was readily agreed it would be highly advantageous to include within our long-range assessment plan a more comprehensive assessment project for Math 205, one that involved multiple instructors and covered more thoroughly the material covered in the course. Problems and rubrics have already been developed in

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12 See Appendix – Math 205 Embedded Assessment Detailed Results
anticipation of such an assessment project, which should make it a fairly straightforward endeavor.

Math 205 Assessment #2

1. Timeframe for the assessment(s).
   - Spring 2008 and Fall 2011

2. Faculty involved.
   - Efren Ruiz and Mitchell Anderson.

3. Student Learning Outcomes Assessed
   - ILO’s – Computational and Analysis
   - CLO – Demonstrate proficiency in using the standard differentiation rules

4. Courses in which the assessment was administered.
   - Math 205

5. Audience (i.e. Math majors, NS majors, non-science majors), levels (i.e., freshmen/soph or junior/senior), and number of participants.
   - Students in these courses are typically freshmen or sophomore NS majors.
   - 50 students from two sections of Efren’s class and 22 students from one section of Mitchell’s class participated in this assessment.

6. Details of the Assessment
   - What type of assessment was administered (Direct or Indirect), and how was the data collected?

   These were direct assessments in which students were administered a single quiz, referred to as a “Gateway” exam\textsuperscript{13} that assessed the students’ ability to use the

\textsuperscript{13} See Appendix – Math 205 Gateway Exam.
standard differentiation rules for the usual families of functions, without simplifying. Students in Mitchell’s course were required to achieve 80% to pass, and students in Efren’s course were required to achieve 100% in order to pass. Students in Mitchell’s class were allowed to re-take a similar exam as many times as they liked, and students in Efren’s course were allowed to retake a similar exam up to 2 times per week for 8 weeks. Students were required in both sections to pass the Gateway exam in order to pass the course. The philosophy here was to encourage students to learn the derivative rules as quickly as possible, since all subsequent topics relied on correct derivatives, and students unable to regularly and efficiently apply the standard differentiation techniques to all the basic families of functions cannot find success in Math 206 and should re-take 205. Re-take exams were always different from previously given exams, but tested the same skill set.

- How was the assessment developed?

Mitchell and Efren developed these assessments independent of input from the department or each other. Nevertheless, these are what would be considered standard derivative problems, covering product rule, quotient rule, and chain rule for all the standard families of functions.

- How was it analyzed? (e.g. What type of scoring rubric was developed, who developed it, and who did the scoring?)

No rubric was necessary. Students had to get everything correct, sans simplification, in order to receive credit. All problems were weighted equally.

7. Results and analysis.

The results for gateway exams differ considerably than most results, since they require students to demonstrate a particular level of achievement.

For Mitchell’s class, all 22 students eventually passed the Gateway exam. 14 of the 22 students passed the gateway exam on the first try. Of the remaining 8, 5 passed on the second try and one on the third. The other two required five attempts. One of these 5-attempt students went for help to the math center on numerous occasions, but seemed to have gotten confused comparing the methods used in class, particularly those for the chain rule, with the help he received there. This is not surprising since some teachers use different techniques and hence the tutors may use techniques different from each instructor. Dr. Anderson finally met with the student for a couple of hours to straighten
him out, and he passed on his next attempt. He worked hard in the class and earned a C for the course. The other 5-attempt student ended up with a D.

For Efren’s class approximately 85% of the students eventually passed the Gateway exam. Unfortunately we no longer have the data on initial success, the actual problems, or the number of subsequent attempts. 15% of the students either failed to pass the Gateway exam or gave up trying. These students did not pass the class.

8. Closing the loop.

The results of this assessment were presented to the department in Fall 2012. Not surprisingly they did not spur any discussion on strengths or weaknesses in differentiation. However, there was a lively discussion on the use of gateway exams, and what was meant by “gateway” in this case, which it was decided was a gateway to 206, and not 205. Some discussion included the possibility of utilizing these exams every time the courses were taught, and there was strong support in this direction. Overall the consensus seemed to be that there were many positive aspects to requiring such a gateway exam, and that making it policy is certainly worth considering. Discussion will continue in this regard.

Math 206 Assessment #1 of 3

1. Timeframe for the assessments.


2. Faculty involved.

- Mitchell Anderson administered the assessments. Brian Wissman and Mitchell Anderson developed the problems and rubrics.

3. Student Learning Outcomes Assessed

- ILO’s – Computational, Analysis, Visual, and Critical Thinking Quantitative Reasoning
- The following Math 206 CLO’s
  - Understand and apply the Fundamental Theorem of Calculus
  - Use the definition of the Definite Integral to set up problem solving models
  - Use the Definite Integral to find volumes by the washer/shell/slicing methods
Identify appropriate methods of anti-differentiation (i.e. u-sub, parts, trig, trig sub) and be able to apply the methods
Solve differential equations
Compute radius and interval of convergence for power series
Compute and manipulate McLaurin Series for sin, cos, exp
Compute [at least] the first few values for a Taylor Polynomial

4. Courses in which the assessment was administered.

- Math 206

5. Audience (i.e. Math majors, NS majors, non-science majors), levels (i.e., freshmen/soph or junior/senior), and number of participants.

- Students in these courses are typically freshmen or sophomore NS majors, with some juniors and seniors.
- 125 students from seven sections participated in this assessment.

6. Details of the Assessment

- What type of assessment was administered (Direct or Indirect), and how was the data collected?

  A direct assessment was employed by embedding 1 – 3 common problems into each exam given throughout each semester.

- How was the assessment developed?

  Brian and Mitchell developed the questions and rubrics\(^{14}\).

- How was it analyzed? (e.g. What type of scoring rubric was developed, who developed it, and who did the scoring?)

  Mitchell scored the results based on the agreed upon rubrics.

7. Results and analysis.

Results for the Math 206 Multi-Year Assessment

\(^{14}\) See Appendix – Math 206 multi-year assessment problems and rubrics.
Note: scores are given as a percentage of the possible raw score

<table>
<thead>
<tr>
<th>Possible Score</th>
<th>n</th>
<th>FTC</th>
<th>Def Int</th>
<th>Vol of Rev</th>
<th>u-sub</th>
<th>parts</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fall ‘09</td>
<td>12</td>
<td>.56</td>
<td>.97</td>
<td>.79</td>
<td>.62</td>
<td>.78</td>
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<td>Spring ’10 (sec. 1)</td>
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<td>.80</td>
<td>.93</td>
<td>.80</td>
<td>.66</td>
<td>.51</td>
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<td>Spring ’10 (sec. 2)</td>
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<td>.69</td>
<td>.79</td>
<td>.83</td>
<td>.75</td>
<td>.69</td>
</tr>
<tr>
<td>Summer ‘11</td>
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<td>.83</td>
<td>.88</td>
<td>.97</td>
<td>.79</td>
<td>.86</td>
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<tr>
<td>Fall ‘11</td>
<td>21</td>
<td>.74</td>
<td>.90</td>
<td>.93</td>
<td>.70</td>
<td>.78</td>
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<tr>
<td>Spring ’12</td>
<td>28</td>
<td>.87</td>
<td>.84</td>
<td>.90</td>
<td>.81</td>
<td>.90</td>
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<tr>
<td>Summer ‘12</td>
<td>16</td>
<td>.65</td>
<td>.89</td>
<td>.84</td>
<td>.90</td>
<td>.85</td>
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<tr>
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<td>.73</td>
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<td>.87</td>
<td>.75</td>
<td>.77</td>
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</table>

Institutional Goals Assessed

<table>
<thead>
<tr>
<th>Quant. Reasoning Translation</th>
<th>Analysis &amp; Calculations</th>
<th>Calculations</th>
<th>Analysis &amp; Visual</th>
<th>Analysis &amp; Calculations</th>
<th>Analysis &amp; Calculations</th>
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<tbody>
<tr>
<td>Competent</td>
<td>Advanced</td>
<td>Advanced</td>
<td>Competent</td>
<td>Competent</td>
<td>Competent</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Trig sub</th>
<th>ODE</th>
<th>Interval of Convergence</th>
<th>Taylor Series</th>
<th>Taylor Series Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>Possible Score</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>Fall ‘09</td>
<td>.70</td>
<td>.40</td>
<td>.62</td>
<td>.50</td>
</tr>
<tr>
<td>Spring ’10 (sec. 1)</td>
<td>.57</td>
<td>NA&lt;sup&gt;15&lt;/sup&gt;</td>
<td>NA</td>
<td>NA</td>
</tr>
<tr>
<td>Spring ’10 (sec. 2)</td>
<td>.67</td>
<td>.70</td>
<td>NA</td>
<td>NA</td>
</tr>
<tr>
<td>Summer ‘11</td>
<td>.88</td>
<td>.73</td>
<td>.69</td>
<td>.66</td>
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<tr>
<td>Fall ‘11</td>
<td>.63</td>
<td>.75</td>
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<td>.62</td>
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<tr>
<td>Spring ‘12</td>
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<td>.70</td>
</tr>
<tr>
<td>Summer ‘12</td>
<td>.83</td>
<td>.78</td>
<td>.83</td>
<td>.58</td>
</tr>
<tr>
<td>Average</td>
<td>.73</td>
<td>.70</td>
<td>.70</td>
<td>.61</td>
</tr>
</tbody>
</table>

<sup>15</sup> NA indicates missing data.
One of the advantages of a multi-year assessment is to be able to look for trends from year to year. While very few of the ten problems indicate a monotonic increase, most show a general trend towards improvement. Fluctuations in the preparedness and quality of students from semester to semester undoubtedly accounts for some of the lack of monotonicity.

Results indicate that for all but one question the average for the students over the six semesters reached a level of Competent or Advanced. More notable is that for the first six questions at least two of the last four semesters achieved advanced ratings, and problems 7 and 8 missed this mark by only 1 or 2 percentage points. This represents a marked improvement over the first three semesters for most of the six questions. Infinite series, represented by the last three questions, clearly remains the most difficult for the students. Dr. Anderson made adjustments over the years in an effort to ensure that students had more time to absorb the material by moving some of his deadlines, so that students could concentrate solely on infinite series when they encountered the subject.

Weaknesses:

i. FTC – the most common misunderstanding for this type of problem is students’ misinterpretation of the integral for negative values. That is to say they compute the negative area incorrectly, computing the area of rectangles or triangles that have an incorrect base length or they use the area that is actually below the negative portion of the graph, drawing the base at the lowest height. It seems they attempt to visually
compute an area, which they know is to be negative, but they ignore the definition of the definite integral as a limit of approximating sums, each rectangle resulting from the height at an x value within the interval where f is negative. So, while they get a negative value, it is computed using the wrong x-interval, where f is actually positive, or they compute the area using the correct x-interval but the wrong heights. The second most common mistake is to break up triangles, the area of which can obviously easily be computed, into many small rectangles, and guessing the values of the many non smaller triangular shapes at the edges.

ii. Definite Integral – this problem is very straightforward. Common mistakes are not antidifferentiating at all, or doing it incorrectly. Another less common mistake is to compute \( F(a) - F(b) \) instead of \( F(b) - F(a) \).

iii. Volumes of Revolution – the scores here are very high in general. The most common mistake is to use the shell method, obviously confusing the two methods.

iv. U-sub – the most common mistakes result from not being organized. In many instances it is difficult to figure out what substitution they make for u, and then in still other cases they are not explicit about changing the dx integral into a du integral, with the important steps missing. Another common mistake is mixing x’s and u’s into the same integral. The most important mistake seems to be that students do not understand that once the substitution is made, if the resultant du integral is not easy, then the process was probably not worthwhile. A sad mistake is when students do not recognize simple u-substitution problems as being such, which indicates they do not truly understand the chain rule or the structure of its results.

v. Parts – the most common mistake with parts is not recognizing that the problem is a parts problem. This problem is a tabular (i.e. repeated) parts problem involving a power of x and a simple sin, cos, or exp. The most common problems here are not correctly anti-differentiating correctly sin 3x, either getting sin 3x/3, or 3cos3x, or missing the negative sign.

vi. Trig sub – this appears to be one of the more difficult techniques for students. First, they have difficulty recognizing it as a trig sub problem. The only explanation for a problem that is so clearly trig sub is that they are confusing the different methods and do not have a good understanding of any of them. One of the more common mistakes students make, once they attempt to apply a trig sub is to pick the wrong trig function, again a silly mistake. Some of the better students even do this. The most obvious weakness associated with this type of problem is lack of organization. There are a lot of steps to this problem and if students are not neat and organized it is easy to lose ones step along the way. Back substitution is the final weakness.

vii. ODE – this problem is very straightforward and the scores should be higher. Students seem to understand that they need to take the appropriate number of derivatives and then plug them all in to see if “it works”, but they are weak in two areas. Some cannot take multiple derivatives. If they use the chain rule the first time it is hard for
some to decide they need to use the product and the chain rule the second time, etc. Then, they have a hard time “showing” that it works. They do not finish the problem properly, stating that the left side of the equation does not equal the right.

viii. Interval of Convergence – applying the ratio test to ratios that involve a variable seems to be the biggest weakness, although by this point it is not uncommon for students to have completely run out of time for these last three problems, not in terms of the test but in terms of being able to study, so they flunk miserably on this part of the exam. Common mistakes are students getting the radius of convergence and not checking the endpoints correctly. It seems they plug everything in correctly but then gloss over the necessary details or analysis of the resulting infinite series.

ix. Taylor Series -- there are two primary mistakes that students make on this particularly simple problem. First, they do not recall the McClaurin series, and second they incorrectly evaluate at $x^2$, which is a precalculus type of skill. Again, it is unfortunate but some students are pretty worn out by this stage of the course. Of course, some students find this problem extremely easy.

x. Taylor Series Coefficients – the first part of this problem is identical in terms of the level of difficulty as the one directly above. A common mistake is differentiating incorrectly. Another is not writing down a series at all. Simple algebra mistakes are common as well. It is interesting to note that this problem probably has the highest standard deviation for its scores. There are a lot of full credit scores for this problem, and unfortunately a lot with a zero. This indicates that if the students are caught up with their work they can certainly master the main concepts here. It is the ones who are behind in the class, and probably in others as well, that struggle to catch up at the end and their infinite series studies suffer, particularly the very last part that we cover, which is the type of problem covered in this problem.

Analysis of this assessment through the Institutional Goals lens

The concepts covered in Math 206 are at the high end of Quantitative Reasoning related concepts, particularly in terms of critical thinking and analytic skills. The problems chosen for this assessment, some of which are the more basic questions encountered in Math 206, still reflect this idea. The Fundamental Theorem of Calculus problem requires students to understand the implications of FTC, and then to apply them in a real world setting, performing multi-level computations and analysis. The volume of revolution problem requires students to visualize in three dimensions and to be able to extract information from that visualization. It also requires students to understand the concept of integration from a “one piece at a time” perspective and to extend that understanding to the implications of the definition of the definite integral as the limit of a sum, thereby allowing the transformation of the sum of individual pieces to a Definite Integral that can easily be computed either by hand or with the use of technology. The integration techniques problems require students to identify and analyze differentiation from the
results end, and to be able to move between differentiation and anti-differentiation within the same problem, combined with faultless precalculus skills. The infinite series questions push students’ analytic and computational skills to their limit, no pun intended. It is extremely pleasing to students and instructors alike to reflect on the students’ abilities to tackle problems of such advanced mathematical sophistication, given the fact that for many it was less than six months prior that they had even learned the simple skill of differentiation. Students achieving competent scores for such Math 206 problems are certainly at the advanced levels intended by the Quantitative Reasoning institutional goals.

8. Closing the loop.

As this assessment progressed Dr. Anderson made adjustments in his teaching, most notably allowing more instructional time for those topics the assessment results indicated students were struggling with, and pointing more frequently to the types of misunderstandings of former students, as noted in the weaknesses above, while still avoiding at all times a possible tendency to teach to the test. A discussion was held with the department on the results in Fall 2012 and a short discussion ensued as to what the faculty considered important learning outcomes for Math 206. It was also suggested that we could use a more comprehensive assessment across all sections of Math 206. This is one of the items included in our revised long range assessment plan.

Math 206 Assessment #2 of 3

1. Timeframe for the assessment.
   - Spring 2012.

2. Faculty involved.
   - Brian Wissman and Mitchell Anderson. (Shuguang Li was involved initially, but did not complete the assessment.)

3. Student Learning Outcomes Assessed
   - ILO’s – Calculations, Analysis, and Critical Thinking (Quantitative Reasoning)
   - The following Math 206 CLO’s
     - Identify appropriate methods of anti-differentiation (i.e. u-sub, parts, trig, trig sub) and be able to apply the methods
     - Compute radius and interval of convergence for power series
Compute and manipulate McLauren Series for sin, cos, exp
Compute [at least] the first few values for a Taylor Polynomial

4. Courses in which the assessment was administered.

- Math 206

5. Audience (i.e. Math majors, NS majors, non-science majors), levels (i.e., freshmen/soph or junior/senior), and number of participants.

- Students in these courses are typically freshmen or sophomore NS majors, with some juniors and seniors.
- 65 students from three sections participated in this assessment, 26 from Mitchell’s single section and the remainder from Brian’s two sections.

6. Details of the Assessment

- What type of assessment was administered (Direct or Indirect), and how was the data collected?

This was a direct assessment in which 6 problems were embedded into two exams, 3 in the techniques of integration exam and 3 in the infinite series exam. (Note: the results for Mitchell’s exams are also included in his multi-year assessment, with the difference being that Brian’s results are included here as well, and Mitchell’s first 3 multi-year problems as well as the ODE problem are omitted here.)

- How was the assessment developed?

Brian and Mitchell developed the problems long ago. These problems were either taken directly from those used in the multi-semester assessment #1 above, or were very similar, with essentially identical rubrics.

- How was it analyzed? (e.g. What type of scoring rubric was developed, who developed it, and who did the scoring?)

Brian and Mitchell developed the rubrics at the same time they identified the problems. They scored their results separately using the common rubric and combined the results.
7. Results and analysis.

<table>
<thead>
<tr>
<th>Possible Score</th>
<th>u-sub</th>
<th>Parts</th>
<th>Trig Sub</th>
<th>Interval of Convergence</th>
<th>Taylor Series</th>
<th>Taylor Series Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sp 2012</td>
<td>.77</td>
<td>.84</td>
<td>.48</td>
<td>.59</td>
<td>.47</td>
<td>.51</td>
</tr>
</tbody>
</table>

ILO’s Analysis & Calculations
QR Translation Competent Advanced Emerging Competent Emerging Emerging

The results here indicate that students are struggling with Trig Sub and most aspects of infinite series.

8. Closing the loop.

The results of this assessment were presented at the same time as those for Math 206 assessment number one above. It was noted that one of the reasons the scores were a bit lower here was that some of the students did not even attempt some of the more difficult problems, such as trig sub, because the exam was a bit long. A discussion ensued as to how to ensure that if you embedded problems for assessment within an exam students would actually attempt it, resulting in a valid assessment of their capabilities for each problem. In the future this will be dealt with. Mitchell noted that he did not experience the same problem, making sure that time was not a critical issue for his exams.

**Math 206 Assessment #3 of 3**

1. Timeframe for the assessment.
   - Spring 2012.

2. Faculty involved.
   - Brian Wissman and Mitchell Anderson.

3. Student Learning Outcomes Assessed
   - ILO’s – Computational, Analysis, Visual, and Critical Thinking (Quantitative Reasoning), and Organization and Structure (Communication)
The following Math 206 CLO’s

- Understand and apply the Fundamental Theorem of Calculus
- Use the definition of the Definite Integral to set up problem solving models
- Understand and apply Integration as a process

The following CLO’s from the lab portion of Math 206

- Determine when it is more appropriate to use technology for Math 206 level problems
- Use technology to compute various approximations to a definite integral (e.g. Left(n), Right(n), and the best available approximation)
- Use technology to compute a sequence of approximations to an applied definite integral problem, and make reasonable conclusions regarding convergence to the answer
- Apply Euler’s method and technology to create a sequence of approximations

4. Courses in which the assessment was administered.

- Math 206

5. Audience (i.e. Math majors, NS majors, non-science majors), levels (i.e., freshmen/soph or junior/senior), and number of participants.

- Students in these courses are typically freshmen or sophomore NS majors, with some juniors and seniors.

- Approximately 27 students from three sections participated in this assessment, 15 from Mitchell’s single section (about half his class) and the remainder from Brian’s two sections.

6. Details of the Assessment

- What type of assessment was administered (Direct or Indirect), and how was the data collected?

This was a direct assessment consisting of a written group project. Mitchell has given two group projects for numerous years in his Math 206 course. Students typically work in groups of three, assigned by the instructor, and hand in a single written report
for which they each receive the same grade. Mitchell has noted a marked decline in
the quality of writing for his projects, which prompted this assessment.

Students in Dr. Anderson’s class were provided, in addition to the group project, a
document entitled Group Project Template\textsuperscript{16} to assist them in understanding how the
report should be written. Students were given a few weeks to work on the projects
and each group turned in a single report, usually 2 – 3 pages in length. Students in
Dr. Wissman’s class worked on their projects individually.

➢ How was the assessment developed?

Brian and Shuguang developed the \textit{Lead Balloon Analysis}\textsuperscript{17} for their project and
Mitchell used his usual \textit{River Skipper}\textsuperscript{18} group project. Brian and Mitchell met and
discussed in detail the intent of the projects.

The purpose of this assessment was multifold.

i. To provide experience working collaboratively on larger endeavors.

ii. To further explore and understand integration as a process.

iii. To improve written communication skills.

➢ How was it analyzed? (e.g. What type of scoring rubric was developed, who
developed it, and who did the scoring?)

A cross section of five of the total ten reports was selected from Mitchell’s class, and
a cross section of four reports was selected from Brian’s class. Shuguang was not
involved in this portion of the assessment. Brian and Mitchell decided to limit the
number of reports to a cross section of each class to reduce the amount of work.
However, they attempted to choose a representative cross section for both projects.

Brian and Mitchell met and after considerable discussion developed a scoring rubric
prior to scoring the projects.\textsuperscript{19} They agreed that these projects assessed both
Quantitative Reasoning ILO’s, including a more than usual level of Critical Thinking,
and Communication ILO’s. From the Quantitative Reasoning ILO’s they identified
both Analysis and Calculations. From the Communication ILO’s they identified
Organization and Structure, and Line of Reasoning. When they attempted to

\textsuperscript{16} See Appendix – Math 206 River Skipper Template
\textsuperscript{17} See Appendix – Math 206 Lead Balloon Analysis
\textsuperscript{18} See Appendix – Math 206 River Skipper
\textsuperscript{19} See Appendix – Math 206 Writing Project Rubric
distinguish which student learning outcomes to assess within each institutional goal it became clear that the Line of Reasoning outcome within Communications was directly tied to the Analysis outcome within Quantitative Reasoning. Students needed to identify the appropriate strategies to mathematically solve the problem, which falls within Analysis, and they needed to communicate them well enough for an outsider to understand, which falls within Line of Reasoning. Thus, they decided to first score Communication based on C1 – Professional Layout, C2 – Level of Prose, C3 – Communication of the Solution, and Quantitative Reasoning based on Q1 – Strategy, and Q2 – Results. They then averaged C1 and C2 (Organization and Structure), C3 and Q1 (Critical Thinking, Analysis, and Line of Reasoning).

After scoring each project individually they met again to compare results and discuss any minor discrepancies.

7. Results and analysis.

The philosophy guiding Dr. Anderson’s projects is that in today’s work environment fellow employees or colleagues often collaborate on large projects and must then present their results. Although Dr. Anderson has received many such reports over the years and has gone to great length to discuss in class the type of reports he expects, the reports he received this semester were unacceptable. He returned them to the students with instructions to re-do them. The poor quality of these “first drafts” prompted the assessment and prompted him to make more explicit his expectations by revising his Group Project Requirements and providing the students with the new version titled Group Project Template. The final drafts were much improved. In the table below A1, A2, etc. refer to group projects from Dr. Anderson’s class and W1, W2, etc. to those in Dr. Wissman’s class.
Math 206 Writing Project Results
Possible scores 1 - 4

<table>
<thead>
<tr>
<th>Individual Reports</th>
<th>CI/C2</th>
<th>C3/Q1</th>
<th>Q2</th>
<th>Avg. Results</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>4</td>
<td>3.75</td>
<td>4</td>
<td>Advanced</td>
</tr>
<tr>
<td>A2</td>
<td>3</td>
<td>2.5</td>
<td>2.5</td>
<td>Approaching Competent</td>
</tr>
<tr>
<td>A3</td>
<td>3.5</td>
<td>2.25</td>
<td>2.5</td>
<td>Approaching Competent</td>
</tr>
<tr>
<td>A4</td>
<td>2.75</td>
<td>2.25</td>
<td>3</td>
<td>Approaching Competent</td>
</tr>
<tr>
<td>A5</td>
<td>2.75</td>
<td>2</td>
<td>1.5</td>
<td>Emerging</td>
</tr>
<tr>
<td>W1</td>
<td>1.5</td>
<td>1.75</td>
<td>2</td>
<td>Emerging</td>
</tr>
<tr>
<td>W2</td>
<td>3</td>
<td>3.25</td>
<td>3</td>
<td>Competent</td>
</tr>
<tr>
<td>W3</td>
<td>3</td>
<td>2.25</td>
<td>2</td>
<td>Emerging</td>
</tr>
<tr>
<td>W4</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>Beginning</td>
</tr>
</tbody>
</table>

Institutional Goals Assessed
- Organization & Structure
- Reasoning & Analysis & Critical Thinking
- Computations

These writing projects are considerably more difficult than other work performed in the class, both in terms of the deeper level of critical thinking and understanding that is required and in terms of the need to communicate the solution in a manner conducive to the reader being able to understand the problem, the strategies for solving the problem, the steps utilized, and the results. For many students Math 206 is their last math class, and our last opportunity to contribute to their education and prepare them for the real world. As such our goal is for most students to demonstrate capabilities at the level of competent or above. While not every student reached this level, more than half reached an average level of Approaching Competency or better.

Weaknesses: The strategy for Mitchell’s project, a non-standard integration problem, requires taking numerical approximations to time and distance traveled and then taking better and better approximations to simulate the convergence of a limit. In many instances students only utilize the first part of this strategy and omit the important second part. In other cases they state that “by the distance formula” something must be true, but it is not always clear if they are referring to the $d = rt$ formula or the distance between two points, both of which need to be used in different places for this problem. This particular weakness is a reflection of deficient communication skills, which show up
throughout the students’ writings. In other cases the entire strategy is wrong, in essence they missed the proverbial boat entirely, in which case their written explanations must necessarily be exceedingly poor as well. Two of the most common strategy mistakes are to try to find average velocity by averaging a bunch of velocities, and the second is to use horizontal distance in place of a linear approximation to distance traveled over a short x-interval. One of the more common communication weaknesses is to include whole paragraphs of irrelevant material, usually explaining how to find the distance the boat traveled. While this may be an interesting process, since they usually break up the river into small pieces and essentially apply the basis for the arc length theorem, which they could have alternatively used in any case, they never use this piece of information elsewhere in the problem.

Note: It should be noted that it is possible to answer this problem by setting up a definite integral, provided the velocity function used does not include a zero. Over the years only one group has ever achieved this, but it appeared at the time that it may have been by trial and error, or they simply tried to figure out a distance formula divided by velocity. The student who achieved this was pretty talented, so he may have understood what he did, but the justification was not very thorough.

In some cases the organization and structure of the writing was very poor. When the organization was adequate students still needed to overcome the challenge that it is very difficult to say just what is true and at just the right level of detail, and most of the students’ papers reflected this difficulty at some level.

8. Closing the loop.

The results of this assessment were presented and discussed in a department meeting. The topic of student writing capabilities and weaknesses were discusses. As a result of this assessment Dr. Anderson implemented four short writing assignments into subsequent Math 206\(^{20}\) classes and one in his Math 205\(^{21}\). He also developed a revised writing template for his group project that was more explicit and better guided the students in presenting a logically flowing and relevant report. The four short writing assignments used a very similar template and were intended to better prepare his students for the more detailed projects. He noted that the student writing improved notably from the first to the fourth, and these assignments helped students with their group projects as well, as the template and writing style became more familiar as the students progressed through the course.

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\(^{20}\) See Appendix – Math 206 Writing Assignments 1 – 4, Overview, and Rubric

\(^{21}\) See Appendix – Math 205 Writing Assignment and Overview
Math 311 Assessment

1. Timeframe for the assessment.
   - Spring 2010 and Spring 2012.

2. Faculty involved.
   - Mitchell and Brian developed the problems and rubrics\(^{22}\). Mitchell conducted the assessments.

3. Student Learning Outcomes Assessed
   - ILO’s – Calculations, Analysis, Visualization and Critical Thinking (Quantitative Reasoning)
   - The following Math 311 CLO’s
     - Demonstrate an understanding of the structure of solutions to linear systems of equations. (eg. Given a single solution to AX = B, find and use the solutions to the homogeneous equation AX = 0 to find all solutions to AX = B.)
     - Apply the subspace theorem to show a subset under the inherited operations forms a vector space.
     - Show that a set of vectors in a vector space is linearly independent/dependent.
     - Find a basis for a [non-standard] vector space.
     - Compute the eigenpairs for a matrix.
   - The following Math Degree PLO’s
     - Demonstrate mastery of the core material found in single and multi-variable Calculus and Linear Algebra.
     - Demonstrate mastery of the core concepts in Abstract Algebra and Real Analysis.
     - Identify, compare, and contrast the fundamental concepts within and across the major areas of mathematics, with particular emphasis on Linear Algebra, Abstract Algebra, and Real Analysis.
     - Use a variety of theorem-proving techniques to prove mathematical results.
     - Demonstrate the abilities to read and articulate mathematics verbally and in writing.

\(^{22}\) See Appendix – Math 311 Problems and Rubrics.
4. Courses in which the assessment was administered.

- Math 311

5. Audience (i.e. Math majors, NS majors, non-science majors), levels (i.e., freshmen/soph or junior/senior), and number of participants.

- Students in these courses are typically Sophomore or Junior Math or Computer Science majors.

- The Spring 2010 assessment involved 16 students and the Spring 2012 assessment involved between 7 and 13 students (this class had a very high drop rate, with surprisingly many students unprepared for the level of work).

6. Details of the Assessment

- What type of assessment was administered (Direct or Indirect), and how was the data collected?

This was a direct assessment in which 1 and 3 problems were embedded into three exams, for a total of 5 questions each semester. The problems used within the more comprehensive set of problems developed by Mitchell and Brian were System of Equations #2, Vector Spaces #2, Linear Independence #2, Basis and Span #1, and Eigenvalues/Eigenvectors #1.

These five problems are representative of the combination of skills and concepts students should be exposed to and learn in Math 311. Showing a set of vectors is linearly independent and finding eigenpairs are primarily computational problems that can be successfully completed without a deep understanding of the underlying concepts. On the other hand, finding all solutions to a non-homogeneous system, given only one solution and a row-equivalent representation of the coefficient matrix, requires students to understand the fundamental structure of solutions to linear systems as well as being able to perform rudimentary algebra. Similarly, proving a subset of vectors forms a vector space is for some students their first foray into theorem proving, and the specific problem used requires familiarity with the concepts of functions that can only be gained through practice from earlier classes. Finally, finding a basis, even for the simplest set of nonstandard vectors, requires a deep understanding of vectors, linear independence, and span, but no computations.
How was the assessment developed?

Brian and Mitchell developed the problems and rubrics.

How was it analyzed? (e.g. What type of scoring rubric was developed, who developed it, and who did the scoring?)

See scoring rubric. Mitchell scored the problems himself, based on the agreed upon rubric.

7. Results and analysis.

This assessment covered student learning outcomes within both Math Dept. Program Goals and Institutional Goals. As indicated by the department’s Course Alignment Matrix, Linear Algebra crosses all program SLO’s. Accordingly, each of the problems utilized within this assessment required students to demonstrate their understanding of at least one standard Linear Algebra Concept. Not surprisingly, this assessment also assessed learning outcomes found within the Quantitative Reasoning institutional goal, including critical thinking, analysis, and computation. The specific program and institutional learning outcomes addressed by each problem are included in the results table below. Critical thinking, while not listed in the institutional goals, is included in each and is not listed for convenience.

Results for the Math 311 Assessment

<table>
<thead>
<tr>
<th></th>
<th>Sys Eqns</th>
<th>Vec Spaces</th>
<th>Lin Ind</th>
<th>Bases/Span</th>
<th>Eigen</th>
</tr>
</thead>
<tbody>
<tr>
<td>Possible Score</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>Spring 2010</td>
<td>1.89</td>
<td>3.88</td>
<td>2.81</td>
<td>2.39</td>
<td>3.25</td>
</tr>
<tr>
<td>Spring 2012</td>
<td>2.15</td>
<td>2.89</td>
<td>2.33</td>
<td>1.11</td>
<td>3.14</td>
</tr>
<tr>
<td>Avg %</td>
<td>.67</td>
<td>.85</td>
<td>.86</td>
<td>.58</td>
<td>.80</td>
</tr>
<tr>
<td>Program SLO’s</td>
<td>1 &amp; 3</td>
<td>1-5</td>
<td>1 &amp; 3</td>
<td>1 &amp; 3</td>
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</tr>
<tr>
<td>Institutional Goals</td>
<td>Analysis &amp; Calculations</td>
<td>Analysis</td>
<td>Analysis &amp; Calculations</td>
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<td>Analysis &amp; Calculations</td>
</tr>
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</tr>
<tr>
<td>Translation</td>
<td>Competent</td>
<td>Advanced</td>
<td>Advanced</td>
<td>Approaching Competence</td>
<td>Advanced</td>
</tr>
</tbody>
</table>

Again, with a multi-year assessment it would be nice to see improvement from year to year. However, in hindsight the Spring 2012 semester was perhaps not the best in which to conduct an assessment. It began with 16 students, ended with 13 registered but only 7 taking the final exam. It was the worse class in terms of sporadic attendance that
Mitchell has experienced in his long career. That said, a comparison of scores in such a small sample is perhaps not all that useful. On the other hand, the scores from Spring 2010 indicate the students had a very good grasp of the material and except for the first problem consistently reached levels that we could describe as competent or advanced. It appears that the students in that class improved dramatically after their first exam, which Mitchell has found typical for Math 311.

Weaknesses: As long as the students have a good understanding of the basic concepts, each of the problems used is very straightforward.

Systems of Equations – The system of equations problem requires students to identify the solution space to a homogeneous system, given by a reduced 3 x 3 matrix, and to combine those results with one solution to an associated non-homogeneous system. Most of the mistakes on this problem result from not understanding the fundamental structure of such systems, not from computational mistakes.

Vector Spaces – This problem requires students to show that the subset of \( P_3 \) with roots at 1 and 2 forms a subspace. The biggest weakness for this problem is not knowing that in order to check closure of addition you need to check whether the sum of two such functions still has roots at 1 and 2. This is a weakness in being able to read mathematics. Students also frequently simply write down necessary results with no work shown, in essence hard wiring the work to indicate it has closure because it has closure. They fail to see, for example, the significance of the difference between addition in \( P_3 \) and addition in the real numbers, each playing a role in this problem.

Linear Independence – This problem is highly computational and really only requires one to take the determinant of a 3 x 3 real matrix and check to see if the result is zero or not. The only obvious weakness for this problem, apart from not being careful with the computations, is not realizing that the determinant is all that is required.

Bases and Span – This problem requires no computation; it only requires students to list a basis for a subspace of \( M_3 \). For a mathematician this is a trivial problem, and as long as the student understands a couple of key concepts, that is still the case. However, students tend to make very serious mistakes on this problem. The biggest mistake, and the most common, is to give as their answer a single or set of vectors that do not even lie in \( M_3 \). This indicates a severe misunderstanding of the underlying concepts. Mitchell has noted to the students that this is the biggest mistake past students have made, and provides numerous examples. Nevertheless, many students continue to make this critical error.
Eigenvalues and Eigenvectors – This problem simply requires students to identify eigenpairs for a 2 x 2 matrix. It is highly computational, but has numerous steps. There does not appear to be any serious misunderstandings associated with this problem. That is not to say that students do not find ways to make critical errors. One aspect that is worrisome is that students do not seem to check their answers, which seems to indicate they do not really understand the concept of Eigenvectors and Eigenvalues.

8. Closing the loop.

These results were presented at a Fall 2012 department meeting and a short discussion ensued. Due to time constraints and the limited nature of the discussion, alternatively faculty members were encouraged to pay particular attention to the weaknesses that were identified and noted in the assessment report and to adjust their teaching accordingly. However, until there is increased participation in assessment, with increased discussion of the results, it is unlikely that this assessment will benefit their teaching. It is the hope of those that have been more fully involved that once the department begins following the assessment plan, with two assessments presented each year, ample time will be allotted for discussion and faculty members will become more engaged in the continuous improvement process.

Math 431 Assessment

1. Timeframe for the assessment.

   Fall 2010 - Spring 2011.

2. Faculty involved.

   - Mitchell Anderson developed the problems and was the instructor. Mitchell, Brian Wissman, Efren Ruiz, and Roberto Pelayo developed the rubrics. Mitchell conducted the assessments, and the team did the scoring.

3. Student Learning Outcomes Assessed

   - ILO’s – Calculations, Analysis, Visualization and Critical Thinking (Quantitative Reasoning)

   - The following Math 431 CLO’s

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23 See Appendix – Math 431-432 Assessment Problems and Rubrics. Also see Appendix – Math 431-432 Full Problem Set for a complete list of the problems presented in the course.

24 See Appendix – Math 431-432 Course Outline
- Demonstrate an understanding of the basic theorems and their implications regarding the topology of the line (and Euclidean n-space if they take 432), sequences and subsequences, compactness, denseness, convergence, continuity, differentiability, the definite integral, cardinality, and metric spaces.
- Utilize a variety of standard theorem proving techniques to construct valid proofs, presented in a logically correct order (e.g. defining variables before using them).
- Construct counter-examples to false statements, and when feasible conjecture hypothesis that would make the statements true.

➢ The following Math Degree PLO’s

- Demonstrate mastery of the core material found in single and multi-variable Calculus and Linear Algebra.
- Demonstrate mastery of the core concepts in Abstract Algebra and Real Analysis.
- Identify, compare, and contrast the fundamental concepts within and across the major areas of mathematics, with particular emphasis on Linear Algebra, Abstract Algebra, and Real Analysis.
- Use a variety of theorem-proving techniques to prove mathematical results.
- Demonstrate the abilities to read and articulate mathematics verbally and in writing.

4. Courses in which the assessment was administered.

➢ Math 431-432

5. Audience (i.e. Math majors, NS majors, non-science majors), levels (i.e., freshmen/soph or junior/senior), and number of participants.

➢ Students in these courses are typically Junior or Senior Math majors.

➢ The Fall 2010 class has 16 students and the Spring 2011 class had 14 students.

6. Details of the Assessment

➢ What type of assessment was administered (Direct or Indirect), and how was the data collected?
This was a direct assessment in which Mitchell gathered a portfolio of 21 proofs presented in class. In this course students were provided definitions and statements that may or may not have been true. Students were required, without the benefit of lecture, discussion, or textbook to identify if a statement was true or false, and in the case it was false to provide a counterexample and perhaps a change of hypothesis to make it true, and in the case it was true to provide a proof. Students then orally presented their results in class. For this academic year Dr. Anderson asked some students to provide him with an electronic copy of their presentations once the class felt they were valid. Dr. Anderson compiled those results into a class portfolio\textsuperscript{25}, including 21 proofs, removed the names, and the four department members scored them based on their common rubrics. They then assembled to discuss their scores and any discrepancies.

Since most of the work had been validated by the rest of the class, most of the portfolio is free from error and hence does not offer the range of results found in other assessments. Nevertheless, it does demonstrate that ILO’s, PLO’s and CLO’s are being met at an advanced level. The 21 proofs ranged from theorems involving the existence of limit points and limits of sequences, to proving that continuous functions over a compact set are uniformly continuous, Cauchy sequences converge, every point in the Cantor set is a limit point of the Cantor set, which is also of measure zero, and integrability existence theorems.

Note: Math 431 – 432 is one of the most difficult course sequences offered by the department, and when offered in this format provides students an unparalleled opportunity for intellectual growth. Since all of the proofs were “accepted” through a peer reviewed presentation process it was expected that most of the scores would be high. This type of assessment appears to be appropriate for this type of class if one wants to assess the level of achievement possible by the class as a whole, particularly in terms of providing students the necessary preparation for graduate school. That being said, in Mitchell’s opinion this particular class did not stand out as exceptional in terms of the quantity and quality of the proofs presented.

\begin{itemize}
  \item How was it analyzed? (e.g. What type of scoring rubric was developed, who developed it, and who did the scoring?)
\end{itemize}

Mitchell, Bob, Brian, and Efren developed the rubrics. Mitchell removed the names from each proof, made copies, and passed them along to the team. Each team member then scored a sample of five of the proofs individually, based on the rubric, and then the team met to discuss the results.

\textsuperscript{25} See Appendix – Math 431 Portfolio
7. Results and analysis.

Overall it was agreed that the level of work demonstrated that the class offered the appropriate level of work in order to meet the PLO numbers 2, 4, and 5. Outcome 2 requires mastery of the core concepts of Abstract Algebra and Analysis. Outcome 3 requires students to use a variety of theorem proving techniques. Outcome 4 requires students to demonstrate the abilities to read and articulate mathematics verbally and in writing. The five theorems the team evaluated included the intermediate value theorem, the convergence of Cauchy sequences, countable number set have measure zero, and the property of bounded variation for non-decreasing functions. Some of the comments the team made dealt with questioning whether or not standard notation was followed for the students (it was), in particular when defining sequences, requirements to include every detail within a proof, and whether or not students were covering every case when more than one case held (e.g. the set in question was finite or infinite).

The team did not find any deficiencies it felt needed addressing, noting that the work indicates a proper level for preparing students for graduate level mathematics. It did however suggest that it would be nice to conduct a more thorough assessment in the future, one that tracked individual student performance and improvement from the beginning of the class to the end, to show value added. The department’s assessment plan has a spot designated for something involving outcomes 2 – 5, which might address this well.

8. Closing the loop.

The results were discussed at a full department meeting. The department echoed the views of the assessment team and agreed that a future more comprehensive assessment might provide valuable information on a more refined level.
1. **Student Learning Outcomes Assessed for the present academic year.**  
   *This section should simply list those SLO’s assessed this year.*

2. **Curriculum Map**  
   *This section should identify how the SLO’s align with our courses. That is, where students encounter opportunities in the curriculum to gain knowledge and skills pertinent to the designated outcomes.*

<table>
<thead>
<tr>
<th>Course Number</th>
<th>SLO#1</th>
<th>SLO#2</th>
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</table>

3. **Assessment Details**  
   *This section should describe how the SLO’s were assessed. (This may be done separately for different assessment efforts.) Be sure to include:*

   - What information/data was collected? (i.e. how was the assessment conducted?)
   - When and by whom?
   - How was it analyzed? (e.g. What type of scoring rubrics were used? Who developed the scoring rubrics? Who did the scoring?)
   - How was it reported?

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1 Adapted from the form used by Phillips Graduate Institute in 2010-2011. Provided by WASC Assessment Workshop, Honolulu, Feb 2012.
4. **Results of Assessment of SLO#1**
   *Attach appendices as necessary.*

5. **Results of Assessment of SLO#2**
   *Attach appendices as necessary.*

6. **Using the results to “Close the Loop”**
   *This section should include:*
   - Describe how the results of the assessments were disseminated and to whom, and the review process used.
   - Discuss how the results will be used (e.g. confirm the SLO was successfully met, and/or how the department will generate strategies for modification).
   - If applicable – discuss program modifications and timeline for implementation.
Math Department Assessment Plan  
February 2008

UHH is currently in a campus-wide assessment effort, and each chair is being asked for three deliverables, due Feb 22. The first is a table showing how each student learning outcome (SLO) is satisfied by students as they move through the major courses, from introduce, to develop and practice, to mastery, where mastery is whatever level we decide is appropriate for that outcome for our majors (see deliverable #1 below). It should be noted that we only need to assess each outcome once every 5 years, for example. It should also be understood that this document is a living document and will change over time. In fact, deliverable number three is a rubric on our progress. Most departments will be at the beginning level. The second deliverable is the assessment plan, included below and on an accompanying spreadsheet.

The purpose of the assessment effort is self-evaluation and reflection of our major program (we will probably want a similar effort for our service mission and gened, but first things first), with the ultimate goal to use the data gathered for improvement (they call this closing the loop). The work below is intended as a point of departure for discussion, but keep in mind we will need to assess at least one of the outcomes next year.

Our student learning outcomes as stated in our last program review, and which are published in the 2004 catalog as well as elsewhere are as follows:

1. A general understanding of the different areas of mathematics and how they interrelate, and the importance of mathematics in a scientifically-oriented society;
2. Classical theorem-proving skills, which include the ability to reason mathematically and to apply the rigor necessary to construct proofs;
3. A refined understanding of the problem-solving process;
4. The ability to independently develop and deliver all pre-college math curriculum, if the professional goal is teaching;
5. A working knowledge of technology appropriate to the field;
6. The skills necessary to
   a. Read, write, translate, and articulate mathematically-related material,
   b. Solve problems using a variety of techniques, including algebraic, numerical, and spatial reasoning through visualization (e.g. graphically),
   c. Make inferences and generalizations.

It has been pointed out that goals can be very general, but student-learning outcomes (SLO) need to be specific and measurable. In general they include action words such as describe, identify, demonstrate, etc. Our outcomes are not bad, but it would help if we were a little more specific. For example, we could simply put in front of each of the above, the word “Demonstrate”. However, I found it quite difficult filling in Deliverable #1, and so I offered the following changes, with explanations and comments following.
**Deliverable #1: Matrix of Program Outcomes and Courses**

The workshop by Dr. Mary Allen on Feb 4-5, 2008, will address the development of program outcomes.

Degree or Program Name: Mathematics (BA)

Program / Department Chair: Dr. Mitchell J. Anderson mitch@hawaii.edu

Revision Date: February 18, 2008

**Math Major – Traditional Track**

Graduating majors should be able to:

Outcome 1 (Knowledge) – Demonstrate mastery of the core material found in single and multi-variable Calculus and Linear Algebra.

Outcome 2 (Knowledge) – Demonstrate mastery of the core concepts in Abstract Algebra and Real Analysis.

Outcome 3 (Comprehension) – Identify, compare, and contrast the fundamental concepts within and across the major areas of mathematics, with particular emphasis on Linear Algebra, Abstract Algebra, and Real Analysis.

Outcome 4 (Reasoning) – Use a variety of theorem-proving techniques to prove mathematical results.

Outcome 5 (Communication) – Demonstrate the abilities to read and articulate mathematics verbally and in writing.

<table>
<thead>
<tr>
<th>Courses for Majors</th>
<th>Require Elective</th>
<th>Outcome 1</th>
<th>Outcome 2</th>
<th>Outcome 3</th>
<th>Outcome 4</th>
<th>Outcome 5</th>
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<tr>
<td>MATH 205,206 - Calculus I-II</td>
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<td>MATH 310 - Discrete Math</td>
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<td>MATH 311 - Intro to Linear Algebra</td>
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<td>Math 317 – Theory of Eqns.</td>
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I = Introduced, D = Developed & Practiced with Feedback, M = Demonstrated at the Mastery
Math Major – Teaching Track

Graduating majors should be able to:

Outcome 1 (Knowledge) – Demonstrate mastery of the core material found in single and multi-variable Calculus and Linear Algebra.

Outcome 2 (Knowledge) – Demonstrate mastery of the core concepts in Abstract Algebra, Real Analysis, Probability, and Statistics.

Outcome 3 (Comprehension) – Identify, compare, and contrast the fundamental concepts within and across the major areas of mathematics, including Linear Algebra, Abstract Algebra, Real Analysis, Geometry, Probability, and Statistics.

Outcome 4 (Reasoning) – Use a variety of theorem-proving techniques to prove mathematical results.

Outcome 5 (Communication) – Demonstrate the abilities to read and articulate mathematics verbally and in writing.

Outcome 6 (Application) – Demonstrate a level of mathematical sophistication consistent with the ability to develop and deliver all pre-college mathematics.

Outcome 7 (Technology) Demonstrate an ability to appropriately use technology in the problem-solving process, including graphing calculators and CAS, and secondarily spreadsheets, statistical software, and proprietary software such as sketchpad.

<table>
<thead>
<tr>
<th>Courses for Majors</th>
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<th>Outcome 1</th>
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<td>MATH 231 - Calculus III</td>
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<td>MATH 310 - Discrete Math</td>
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The SLO for the traditional track may seem skewed towards Algebra and Analysis, but those are the only two senior level sequences that are required for the traditional track, and if we were to include other courses such as Differential Equations, Geometry or Prob/Stats (two of which show up in the teaching track) the assessment matrix would indicate that either those are not important courses or that we should make them required. This of course should be discussed. This is the problem with modeling and technology as well. They are lower level skills, don’t appear at all in the required senior level courses as major objectives, and very little in 310 or 311, and as such I thought they should simply be included as part of outcome 1.

**Discussion of the SLO**

**Outcome 1:** Seems reasonable to want to require that our majors are familiar with differentiation, the standard anti-differentiation techniques, linear approximation, optimization, Taylor Series, multi-variable Calculus, row reduction, co-factor expansion, bases, and eigenvalues/eigenvectors. The list will complete itself during the assessment planning process, through department discussion. Some modeling and technology should also be included here, e.g. the ability to model using symbolic, numeric, and graphical, and appropriately use technology.

**Outcome 2:** The kinds of things we might be interested in here are, for example, compute a particular quotient group, use the idea of basis to solve a linear differential equation when provided with a particular solution and enough linearly independent solutions to the homogeneous, identify the image of a linear transformation using eigenvectors, use compactness and continuity to guarantee the existence of optimal solutions, etc.

**Outcome 3:** Graduating students should know the major concepts in Algebra and Analysis and be able to distinguish between the two. In particular, they should understand that in Algebra the goal is to understand the algebraic structure, while in Analysis the goal is more topological and involves the concept of closeness (limit). It may be the case that Outcome 3 should be included in Outcome 2, but again discussion is needed here.

**Outcome 4:** Probably the most important, and the one with the most parts. For this outcome we would want them to a) (knowledge) know the different techniques such as induction, direct, contrapositive, constructive, and contradiction; b) (knowledge) be able to provide proofs to intermediate-difficult theorems; c) (synthesis) be able to independently provide proofs to previously unfamiliar but basic theorems; d) (evaluation) decide whether or not an argument is valid; e) (analysis) be able to identify and explain which part of a desired proof is the most difficult to overcome; and f) (analysis and synthesis) be able to decide when it is appropriate to use counter-example, and then provide one.

**Outcome 5:** There should be much opportunity in the senior level courses to assess this, but the details need to be worked out.
Further Discussion on Teaching Track SLO’s:

Outcome 6: this is what they will be expected to do once they enter their teaching careers. Topics might include, for example, conic sections and topics in Algebra III, those things that they might be required to teach, but are not covered in our major. In the old days we would have put geometry into this list, but now we cover that.

Outcome 7: for future teachers this is critical. There are two main points, the word appropriately cannot be overemphasized, and the other is familiarity with the technology. They should master this as tutors in the math lab. For those not tutoring we might have a problem, although everyone needs to do in-service learning. We may want to make the tutoring mandatory.

Deliverable # 2 – Assessment Plan (Please see attachment for schedule summary.)

Outcome 1 (Basic Knowledge). The assessment plan for this objective is multifaceted. Embedded assignments and/or course activities will be used in two sections of Math 206, same semester, different instructors, one section of Math 232, and two sections of Math 311 in subsequent semesters, different instructors. This formative assessment will occur every 5 years, beginning 2008-2009. Then, as part of an exit exam, seniors will be tested every other year beginning 2009-2010. The exit exam will also test Outcomes 2 and 3.

Outcome 2 (Advanced Knowledge): An exit exam will be given to seniors every other year beginning 2009-2010. Embedded assignments and/or course activities will be included in Math 431 and Math 454 in subsequent years, and in Math 421-422 (for teaching track majors), and will follow a 5-year cycle beginning 2009-2010.

Outcome 3 (Comprehension): This outcome will be included in the exit exam. Embedded assignments and/or course activities will be included in Math 431 and Math 454 in subsequent years, as well as two sections of Math 311 in subsequent semesters, and for teaching-track majors Math 421-422 and 441, and will follow the same cycle as assessment for outcome 2.

Outcome 4 (Reasoning): Embedded assignments and/or course activities will be included in Math 431 and Math 454 in subsequent years, and will follow a 5-year cycle beginning 2011-2012.

Outcome 5 (Communication): Embedded assignments and/or course activities will be included in two sections of Math 206, different instructors (formative), and in Math 431 and Math 454 in subsequent years, and Math 441 (teaching track), and will follow the same assessment schedule as outcome 4.

Outcome 6 (Application): Assessment will be included, for teaching track students only, in the exit exam. We will also track the success of our students taking the Praxis exams, as well as working with the Education department to develop embedded assignments during their teaching credential program. This last activity will follow a 5-year cycle beginning 2010-2011.
Outcome 7 (Technology): Part of this will be included in the exit exam. This one needs further discussion. We could include embedded assignments in 441 and 422, and we obviously could include 206, including the lab, which would be formative, and/or we could require something of the tutors during 496, and/or we could develop a course around this subject, and/or we could work with the ed dept to include something during their student teaching, etc. This assessment activity is scheduled on a 5-year cycle beginning 2013-14.

Status/Progress/Results

The status of the entire assessment plan is “in discussion”. However, the first assessment activity is scheduled for 2008-2009. Under current consideration: Do we need the formative assessment for outcome 1? Should the exit exam be given annually? Should we try using the ETS exit exam, make up our own, or use a combination of both? Is the assessment schedule appropriate or should we change the order? What is the best way to assess outcomes 6 and 7? Should we include other courses in the assessment, such as Math 300-301?

**Deliverable #3: WASC Rubrics (Self-Assessment Checklist)**

Degree or Program Name: __Mathematics__ (BA)

Program / Department Chair: __Dr. Mitchell J. Anderson mitch@hawaii.edu__

Revision Date: __January 7, 2008__

Check the status of your program’s learning outcomes for each WASC criteria below.

<table>
<thead>
<tr>
<th>Comprehensive List:</th>
<th>Initial</th>
<th>X Emerging</th>
<th>X Developed</th>
<th>____ Highly Developed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Assessable Outcomes:</td>
<td>X Initial</td>
<td>Emerging</td>
<td>Developed</td>
<td>____ Highly Developed</td>
</tr>
<tr>
<td>Alignment:</td>
<td>____ Initial</td>
<td>X Emerging</td>
<td>X Developed</td>
<td>____ Highly Developed</td>
</tr>
<tr>
<td>Assessment Planning:</td>
<td>____ Initial</td>
<td>X Emerging</td>
<td>X Developed</td>
<td>____ Highly Developed</td>
</tr>
<tr>
<td>The Student Experience:</td>
<td>X Initial</td>
<td>X Emerging</td>
<td>____ Developed</td>
<td>____ Highly Developed</td>
</tr>
</tbody>
</table>

See below for explanation of this deliverable.
Rubric for Assessing the Quality of Academic Program Learning Outcomes—August 10, 2007 Draft

<table>
<thead>
<tr>
<th>Criterion</th>
<th>Initial</th>
<th>Emerging</th>
<th>Developed</th>
<th>Highly Developed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Comprehensive List</td>
<td>The list of outcomes is problematic, e.g., very incomplete, overly detailed, inappropriate, disorganized. It may include only discipline-specific learning, ignoring relevant institution-wide learning. The list may confuse learning processes (e.g., doing an internship) with learning outcomes (e.g., application of theory to real-world problems).</td>
<td>The list includes reasonable outcomes but does not specify expectations for the program as a whole. Relevant institution-wide learning outcomes and national disciplinary standards may be ignored. Distinctions between expectations for undergraduate and graduate programs may be unclear.</td>
<td>The list is a well-organized set of reasonable outcomes that focus on the key knowledge, skills, and values students learn in the program. It includes relevant institution-wide outcomes (e.g., communication or critical thinking skills). Outcomes are appropriate for the level (undergraduate vs. graduate) and national disciplinary standards have been considered.</td>
<td>The list is reasonable, appropriate, and comprehensive, with clear distinctions between undergraduate and graduate expectations. If applicable, national disciplinary standards have been considered. Faculty have agreed on explicit criteria for assessing students' level of mastery of each outcome.</td>
</tr>
<tr>
<td>Assesable Outcomes</td>
<td>Outcome statements do not identify what students can do to demonstrate learning. Statements such as “Students understand scientific method” do not specify how understanding can be demonstrated and assessed.</td>
<td>Most of the outcomes indicate how students can demonstrate their learning.</td>
<td>Each outcome describes how students can demonstrate learning, e.g., “Graduates can write reports in APA style” or “Graduates can make original contributions to biological knowledge.”</td>
<td>Outcomes describe how students can demonstrate their learning. Faculty have agreed on explicit criteria statements, such as rubrics, and have identified examples of student performance at varying levels for each outcome.</td>
</tr>
<tr>
<td>Alignment</td>
<td>There is no clear relationship between the outcomes and the curriculum that students experience.</td>
<td>Students appear to be given reasonable opportunities to develop the outcomes in the required curriculum.</td>
<td>The curriculum is designed to provide opportunities for students to learn and to develop increasing sophistication with respect to each outcome. This design may be summarized in a curriculum map.</td>
<td>Pedagogy, grading, the curriculum, relevant student support services, and co-curriculum are explicitly and intentionally aligned with each outcome. Curriculum map indicates increasing levels of proficiency.</td>
</tr>
<tr>
<td>Assessment Planning</td>
<td>There is no formal plan for assessing each outcome.</td>
<td>The program relies on short-term planning, such as selecting which outcome(s) to assess in the current year.</td>
<td>The program has a reasonable, multi-year assessment plan that identifies when each outcome will be assessed. The plan may explicitly include analysis and implementation of improvements.</td>
<td>The program has a fully-articulated, sustainable, multi-year assessment plan that describes when and how each outcome will be assessed and how improvements based on findings will be implemented. The plan is routinely examined and revised, as needed.</td>
</tr>
<tr>
<td>The Student Experience</td>
<td>Students know little or nothing about the overall outcomes of the program. Communication of outcomes to students, e.g., in syllabus or catalog, is spotty or nonexistent.</td>
<td>Students have some knowledge of program outcomes. Communication is occasional and informal, left to individual faculty or advisors.</td>
<td>Students have a good grasp of program outcomes. They may use them to guide their own learning. Outcomes are included in most syllabi and are readily available in the catalog, on the web page, and elsewhere.</td>
<td>Students are well-acquainted with program outcomes and may participate in creation and use of rubrics. They are skilled at self-assessing in relation to the outcomes and levels of performance. Program policy calls for inclusion of outcomes in all course syllabi, and they are readily available in other program documents.</td>
</tr>
</tbody>
</table>
Math Department Assessment Plan

2013-2019

What follows is the mathematics department’s tentative long range assessment plan at the time of the 2013 Program Review. The assessment plan is designed to address the six major pieces of our instructional mission.

1. **Developmental** – preparing underprepared students for success in STEM disciplines and others that require more than nominal mathematics. Our developmental mission is usually satisfied through Math 103 and 104F.
2. Mathematics for **non-science** majors. This part of our mission is usually satisfied through Math 100, 121, and 115.
3. **STEM** mathematics (the Calculus two-year sequence)
4. Preparing students for the **Transition** from the Calculus to higher level mathematics. This part of our mission is usually satisfied through courses such as Math 310, 311, 314, and 317.
5. **Teaching** Track majors. Preparing our teaching track majors for a successful career in secondary teaching is accomplished through a variety of courses such as Math 421-22, 431, 441, 454, 496, and perhaps an as-yet undeveloped capstone experience.
6. **Traditional** Track majors. Preparing our traditional track majors for success in graduate school or mathematics-based careers is accomplished through a variety of courses such as Math 431-32, 454-55, topics courses, and perhaps an as-yet undeveloped capstone experience.

Our plan is to have at least two faculty members take responsibility for leading a meaningful assessment in each of the six instructional areas. The faculty members will not necessarily be required to concurrently teach the courses in which the assessment occurs, if in fact it occurs in the classroom, but will be required to lead the effort, identify the student learning outcomes that will be assessed, design the assessment mechanisms (e.g. embedded problems, rubrics, procedures, etc.) provide leadership and assistance as necessary, and formulate the annual assessment report. Dr. Anderson will be available as a consultant to all efforts, and will attempt to ensure that the assessments are relevant and as a whole provide a comprehensive view of how well we achieve our instructional mission in terms of student success.

What follows is a tentative schedule for the assessment efforts. Planning and design should occur during the year prior to each assessment; and it is anticipated that some efforts may continue for more than a single year or may be adjusted and repeated at a later time. Undoubtedly some faculty members will want to assess the effectiveness of their efforts at
closing the loop in terms of improving student learning. Thus, this assessment schedule is tentative in its most literal sense.

Assessment Schedule

<table>
<thead>
<tr>
<th>Year</th>
<th>Mission Area</th>
<th>Faculty</th>
</tr>
</thead>
<tbody>
<tr>
<td>2013-14</td>
<td>STEM – Calculus</td>
<td>Wissman, Lazarevic, New “Calculus Lab” Instructor</td>
</tr>
<tr>
<td>2014-15</td>
<td>Traditional Track</td>
<td>Pelayo, Ruiz, Wissman</td>
</tr>
<tr>
<td>2015-16</td>
<td>Developmental</td>
<td>New Developmental Instructor, with additional assistance from a TBD volunteer</td>
</tr>
<tr>
<td>2016-17</td>
<td>Transition</td>
<td>Figueroa-Centeno, Li</td>
</tr>
<tr>
<td>2017-18</td>
<td>Teaching Track</td>
<td>Ivanova, Bernstein</td>
</tr>
<tr>
<td>2018-19</td>
<td>Non-Science</td>
<td>Webb, Bernstein, Lazarevic</td>
</tr>
<tr>
<td></td>
<td>Line of Reasoning</td>
<td>Organization and Structure</td>
</tr>
<tr>
<td>------</td>
<td>----------------------------------------------------------------------------------</td>
<td>---------------------------------------------------------------------------------------------</td>
</tr>
<tr>
<td>4</td>
<td>• Well-defined thesis that is supported by coherent and relevant arguments</td>
<td>• Organization is logical, well-planned, and organized; structure enhances the message or argument</td>
</tr>
<tr>
<td></td>
<td>• Ideas and main points are based on logical and rational deductions</td>
<td>• Paragraphs are well-developed, and paragraph breaks enhance the main points</td>
</tr>
<tr>
<td>3</td>
<td>• Identifiable thesis with some gaps or inconsistencies in reasoning</td>
<td>• Some organizational problems evident</td>
</tr>
<tr>
<td></td>
<td>• Some ideas or main points may not be fully integrated into the presentation and essay</td>
<td>• Paragraphs are developed but exhibit a few inappropriate breaks, or transitions between paragraphs are awkward</td>
</tr>
<tr>
<td>2</td>
<td>• Thesis is weak, unclear or too broad for assignment, but has some relevance to the body of essay or presentation</td>
<td>• Some attempt at organization but essay or presentation suffers from gaps in logic</td>
</tr>
<tr>
<td></td>
<td>• Ideas or main points are based on unsubstantiated reasons or speculations</td>
<td>• Paragraphs are underdeveloped and/or Transitions are highly problematic</td>
</tr>
<tr>
<td>1</td>
<td>• No discernable thesis</td>
<td>• Lack of organization (line of reasoning is absent)</td>
</tr>
<tr>
<td></td>
<td>• Ideas or main points of the presentation or essay are unclear, unsubstantiated, or unrelated</td>
<td>• Transitions between paragraphs are non-existent</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### Appendix A-Institutional Goals for Quantitative Reasoning and Scientific Inquiry

<table>
<thead>
<tr>
<th>Level</th>
<th>Analysis*</th>
<th>Calculations</th>
<th>Visual Representations of Data and Information*</th>
<th>Scientific Methodology</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 (Advanced)</td>
<td>• Demonstrates advanced reasoning based on quantifiable information; judgments and conclusions are exceptionally insightful</td>
<td>• Accurately completes calculations for the assignment and presents results clearly and concisely • Chooses appropriate formulas or symbolic models to solve problems and justify choices</td>
<td>• Produces highly effective visual representations of data (e.g. tables) or concepts (e.g. graphs)</td>
<td>• Skillfully and precisely engages in the 6 steps needed in undertaking a science-based approach to gathering and interpreting evidence 1. Identify problem 2. Formulate a hypothesis 3. Design a project to test hypothesis 4. Collect data 5. Analyze data 6. Draw conclusions based on data • Exhibits highly accurate and exhaustive analysis of data • Produces work that contributes to the field</td>
</tr>
<tr>
<td>3 (Competent)</td>
<td>• Demonstrates competent reasoning based on quantifiable information; judgments and conclusions are adequate and reasonable</td>
<td>• Calculations are completed and largely successful • Chooses appropriate formulas or symbolic models to solve problems and justify choices</td>
<td>• Produces competent visual representations of data</td>
<td>• Engages in all 6 steps needed in undertaking a science-based approach to gathering and interpreting data • Produces an analysis of data • Produces work that meets the requirements of the assignments/course</td>
</tr>
<tr>
<td>2 (Emerging)</td>
<td>• Demonstrates emerging reasoning based on quantifiable information as exhibited by difficulty in formulating judgments or drawing conclusions</td>
<td>• Calculations contain multiple errors • May not choose the most appropriate or effective formula • May exhibit some problems justifying choices</td>
<td>• Visual representations may reflect minor flaws or inaccuracies</td>
<td>• Engages in the 6 steps but may exhibit problems with a few • Analysis of data may reflect minor inaccuracies of observation • Work may not fully satisfy the requirements of the assignment/course</td>
</tr>
<tr>
<td>1 (Beginning)</td>
<td>• Demonstrates beginning reasoning based on quantifiable information as exhibited by difficulty understanding what constitutes quantifiable information, inability to formulate reasonable judgments and/or drawing reasonable conclusions.</td>
<td>• Calculations may be unsuccessful or incomplete • Does not appear to understand the parameters of the appropriate formula • Is unable to select the right formula for the problem (decision-making unclear)</td>
<td>• The method for visually presenting information or concepts is highly inaccurate or imprecise</td>
<td>• Exhibits problems in many if not most of the steps required for the scientific process • Analysis of data is incomplete, inaccurate, or absent • Work does not satisfy the requirements of the assignment/course</td>
</tr>
</tbody>
</table>
Appendix A-Institutional Goals for Quantitative Reasoning and Scientific Inquiry
Note: Math 104 is composed primarily of the material covered in Math 104F and 104G.

Catalog Description Math 104

Math 104 is an intensive one semester focus on the material covered in the sequence Math 104F-104G. A student may not receive credit for both Math 104 and Math 104F-104G. Pre: B+ or better in Math 103, or C in Math 104F, or a Math Placement Score greater than 29.

Catalog Description Math 104F

Functions and relations; polynomial and rational functions; exponential and logarithmic functions. Pre: C or better in Math 103 or Math Placement Score greater than 19.

Catalog Description Math 104G

Trigonometric functions; analytic trigonometry; analytic geometry. Pre: C or better in Math 104F, or Math Placement Score greater than 39

Previous Courses Assumed – Two years of high school algebra

Course Goals

The goal of Math 104 is to prepare students for first-year Calculus. Helping students gain proficiency in their understanding and ability to utilize real-valued functions, the primary tool in Calculus, accomplishes this goal. Students are presented a broad set of ‘function tools’, including a general understanding of function properties together with a ‘library’ of commonly used functions. It is intended that students become skilled at recognizing the different families of functions and the primary properties that set each apart, are able to apply the general function properties to each type of function, and are able to use the special set of algebraic skills associated with each. Students are also expected to become adept in utilizing and interpreting the results from graphing calculators, as an important investigative tool.

Note: Given the importance of these courses to success in Calculus, it is expected that the final exams for each of these courses be cumulative.

Learning Outcomes Objectives

Students should be able to:
• Identify domains for the standard functions (power functions, including polynomial and root, rational, exponential, and trig), both symbolically and graphically
• Apply function notation, particularly for the composition of functions
• Translate functions graphically
• Provide common sense interpretations of function inverse
• Solve all equations graphically
• Recognize the standard functions and be able to identify what sets each apart from the others
• Apply the Remainder/Factor theorem and Rational Root Theorem to find roots and factor polynomials
• Identify horizontal and vertical asymptotes for rational functions
• Demonstrate and understanding of the “doubling” property of the exponential function
• Solve exponential and logarithmic equations
• Model exponential growth and decay problems
• Identify without the use of external tools the six trigonometric function values of all values that are integer increments of $\frac{\pi}{6}$, $\frac{\pi}{4}$.
• Utilize both the right triangle and unit circle definitions of sin and cos, and use the reciprocal definitions for the other 4 trigonometric functions
• Solve triangles
• Derive two Pythagorean identities from $\sin^2 + \cos^2 = 1$.
• Use the addition formulas to derive double angle formulas, reduction formulas, and product to sum formulas
• Utilize reciprocal identities, Pythagorean identities, and double angle identities, along with algebraic techniques to simplify trig expressions and verify identities
• Solve trig equations, both symbolically and graphically

Optional for Math 104, but should be covered in Math 104G
Students should be able to:
• Apply DeMoivre’s formula, either symbolically or conceptually, to simplify integer powers of complex numbers that are multiples of $\frac{\pi}{6}$, $\frac{\pi}{4}$
• Plot polar coordinates and plot polar functions
• Plot parametric functions

Use of Technology

Students are expected to have a graphing calculator available in class, for homework, and for exams unless otherwise stated. Instructors are expected to utilize this technology where appropriate, as Calculus instructors will expect incoming students to be proficient in their use. The suggested model is the TI-83 or 84, although any calculator that has the capability to graph functions and find roots should be acceptable. Instructors should be prepared to assist students in the use of this model, but cannot be expected to be
proficient in other models. As for hand-held computers that perform symbolic computations such as for example the TI-89 and TI-92, their use is at the discretion of the instructor. It is understood that some topics require the use of graphing calculators (e.g. domain type questions) while others do not (e.g. analytic trig). The philosophy of the UHH Math department is that this technology is essential to the students’ learning of the Calculus, is highly consistent with contemporary trends, and that pre-Calculus is the appropriate level to begin students’ training in its use.
Topics Covered

Math 104F
- General properties of functions (domain, image, graphs, transformations, inverses, roots, composition)
- Polynomial, Power, and Rational Functions
- Exponential and Logarithmic Functions

Math 104G
- Trigonometric Functions (unit circle trig definitions, right triangle trig, analytic trig)
- Parametric and Polar Functions (time permitting in 104)

Course Objectives

The course objectives for each topic below generally fall into three categories
- Understand and apply fundamental concepts
- Gain skills and techniques to perform symbolic manipulation
- Utilize appropriate technology (e.g. graphing calculator)

General Properties of Functions (6 weeks Math 104F, 3.5 weeks Math 104)

Domains – Ability to reduce composition-based domain problems down to the problem of solving an algebraic equation or inequality. Example: Find the domain for \( \sqrt{x^2 - 1} \) reduces down to solving \( x^2 - 1 \geq 0 \). These problems are then solved either symbolically or graphically. Students need to recognize, for example, that finding the domain for \( \frac{1}{x-1} \) requires the use of technology and that the solution is found not by graphing this function, but by graphing the inside function and finding its roots.

Image – Ability to distinguish the difference between domain and image, and to identify the image for the basic power and radical functions, and later in the course for the basic exponential, logarithmic, and trigonometric functions. In terms of utilizing the graphing calculator, it is essential students become proficient in ‘locating’ the graph on the graphing utility in terms of setting the appropriate window sizes.

Graphs – Ability to represent information provided verbally, symbolically, or numerically on a graph, and the ability to conclude information from a graph.

Translations – Given the graph of a function, say \( f \), students should be able to recognize how the graph is translated when a new function, say \( g \), is defined as \( g(x) = f(x) + c \), \( g(x) = c \cdot f(x) \), \( g(x) = f(c \cdot x) \), \( g(x) = f(x-c) \), or any combination thereof. This capability should be extended to each of the standard functions, in particular exponential, logarithmic, and trigonometric.
**Inverses** – Ability to understand, primarily from a conceptual perspective, the true meaning of inverse. Example: Given that water rises in a container and \( h \) represents the height at time \( t \), students should be able to clearly distinguish the difference between \( h(5) \) and \( h^{-1}(5) \). It is also important to understand the relationship between the graphs of \( f \) and \( f^{-1} \) and that \( f \circ f^{-1} \) is the identity function.

**Roots** – Understanding the connection between finding roots and solving equations; ability to utilize the graphing calculator to locate numerical roots. (More on roots in polynomials below.)

**Composition** – Ability to express complex functions as the composition of basic functions. Students should achieve a clear understanding of the relationship between composition and domain. A desirable objective is for students to be able to visualize the composition of two functions (e.g. \( \sin \frac{1}{x} \)).

**Polynomial, Power, and Rational Functions (4 weeks Math 104F, 2 weeks Math 104)**

**Concepts** – Ability to visualize the behavior (i.e. general graph) of all basic power functions \( f(x) = x^n \), where \( n \) is a number other than zero. Extend this understanding to an ability to graph (without graphing utility) polynomial functions, provided the roots and their multiplicity are known. Extend this yet further to an ability to graph (without graphing utility) rational functions, including finding all roots, vertical and horizontal asymptotes. Students should also be able to find possible equations when provided graphs of polynomial and rational functions. Students should understand the relationship between the degrees of the numerator and denominator polynomials as they apply to the horizontal asymptote. A desirable objective is for students to be able to identify non-horizontal asymptotes (including non-slant ones) and to understand the comparison between the rational function and the limiting polynomial. An understanding of the main theorems such as Rational Root, Factor/Remainder Theorem, ‘complex roots come in pairs’, is also desirable.

**Manipulation Skills** – Ability to recursively apply the rational root theorem to identify possible rational roots of polynomials (synthetic division is optional), and then to use division to factor until the polynomial is reduced to a quadratic, in which case exact values can be found. Students should then be able to provide possible graphs that agree at all intercepts and have correct behavior as they approach infinity.

**Technology** – Since the emphasis in this chapter is on understanding the behavior of polynomial related functions, the graphing calculator plays a diminished role, and is intended primarily to check answers and to assist students when they get ‘stuck’. However, the calculator plays an important role in demonstrating that all
polynomials and rational functions resemble $x^n$ relatively (i.e. if the window is chosen appropriately), where $n$ is a non-negative integer.

**Exponential and Logarithmic Functions (5 weeks Math 104F, 2.5 weeks Math 104)**

*Concepts* – A clear understanding of the ‘doubling’ property (% growth in general) of exponential functions as demonstrated by an ability to find ‘$a$’ and ‘$c$’ in $f(x) = c*a^x$, provided 2 points are given whose $x$ coordinates are either 1 or 2 units apart and ‘$a$’ is a ‘nice’ number (e.g. 1.1, 1.5, 2, 3, 10, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{10}$). Students should be able to ‘check’ whether or not a function, given in table form, might be exponential or not, and clearly understand the difference between linear data and exponential data. Students should be able to recognize the graphs of exponential and logarithmic functions, with particular attention to the relationships between their respective domains and images. Special care should be taken to ensure that students are able to extend the domain skills from the first part of the course to exponential and log functions. Desirable concepts include 1) the results of multiplying polynomial functions by $e^{-x}$; 2) comparing polynomial and exponential functions such as $x^3$ and $3^x - 2$. Finally, students should understand that an exponential function base $a$ is simply a horizontal stretch translation of the exponential function base $e$, so that only one base is necessary. Students should also be exposed to why $e$ might be considered ‘natural’.

*Manipulation Skills* – Ability to algebraically solve for ‘$a$’ and ‘$c$’, given two points on an exponential function, regardless of the values of ‘$a$’ and ‘$c$’, including the standard application problems of population growth, compound interest, and radioactive decay. Students should be able to solve exponential and logarithmic equations, and to apply the properties of log, particularly as they are used in Calculus for logarithmic differentiation. Desirable skills include the ability to algebraically move from one base to another.

*Technology* – It is intended that the graphing calculators be used to help demonstrate the concepts. For example, consider $f(x) = 2^x$, which doubles going from $x$ to $x+1$. Tracing along this function, say at $x = 1.3124$ we get a numeric value for $y$, and moving to 2.3124 we see that $y$ doubles. Similarly, the calculator can be used to demonstrate the other concepts mentioned above. Care must be taken in realizing that the graphing calculator’s regression capabilities can provide the students too much help during this critical introduction to exponential functions, so the use of calculators should be balanced.

**Trigonometric Functions (beginning of Math 104G, or halfway through 104)**

**Unit Circle and Right Triangle Trig (5 weeks Math 104G, 3 weeks Math 104)**

*Concepts* – The relationship between the unit circle and the trig functions (i.e. students must be able to state the definition of sin and cos in terms of the unit circle). Students should be able to state the reciprocal definitions for each of the
other 4 trig functions in terms of sin and cos, and should be able to recognize the
graphs for all the standard trig functions. Students should be able to apply 104F
translation techniques to sin and cos and identify the effects of translations in
terms of changes to amplitude, period/frequency, line of symmetry, and phase
shift. They should be able to find the equation for a sin/cos function given
appropriate information (e.g. verbally or graphically).

Manipulation Skills – Ability to locate standard angles (i.e. all multiples of \( \pi/6 \)
and \( \pi/4 \)) on the unit circle and provide exact values for the 6 trig functions.

Emphasis should be on memorizing the three numbers \( \frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}} \), next
locating points on the unit circle and deciding which of these, or 0 or 1, should be
used for sin/cos, and then using the reciprocal definitions to find the other four
values. Students should be able to apply either the Pythagorean identities or
preferably the right triangle definitions for each of the trig functions to find the
other 5 trig function values (exact) when given one value and the quadrant or
equivalent information. This technique is used in Calculus 2 in association with
trig substitutions. Of secondary consideration is the ability to solve any triangle,
given three of the six pieces of information, one of which is a side. This requires
the use of the Law of Sines/Cosines and is generally not included on the
cumulative final exam.

Analytic Trig (8 weeks Math 104G, 5 weeks Math 104)

Manipulation Skills – Ability to DERIVE the Pythagorean identities involving
tan and cot, beginning with the standard identity. Using the addition formulas for
sin/cos (which should be memorized), students should be able to DERIVE

- the double angle formulas for sin/cos (all 3 versions for cos)
- extend the double angle formulas to derive identities for \( \sin^2 \) and \( \cos^2 \)
- the product-to-sum transformations for products of sin and cos
- reduction formulas (e.g. reduce tan (\( \theta - \pi/3 \)) to a trig function involving \( \theta \) only)

Students should become familiar enough with manipulating trig functions so as to
be able to easily ‘prove’ trig identities, including ones that contain double angles
on one side of the equation and single angles on the other. Finally, students should
be able to solve standard trig equations (i.e. those that reduce down to single trig
functions of a single angle) and equations involving trig functions of multiples of
\( \theta \) (e.g. those reducing to a form such as \( \cos 3\theta = \frac{\sqrt{3}}{2} \)). A desirable skill is for
students to be able to solve problems such as: Given \( \sin \theta = x \), find \( \sin 2\theta \) in terms
of \( x \). (Again this comes up with trig substitutions in Calculus 2.)
Parametric and Polar Functions (2 weeks Math 104G, 1 week optional Math 104)

Concepts – Ability to understand the relationship between r-θ Cartesian graphs and their corresponding polar representation. Similarly students should understand the relationship between the two coordinate Cartesian graphs of a function given parametrically (i.e. x vs. t and y vs. t) and its [combined] graph (i.e. the graph of f(t) = (x(t), y(t)). Of primary interest for parametric functions is the ability, when possible, to identify a recognizable x-y relation between the two coordinate functions, x(t) and y(t).

Skills – Ability to plot points given in polar coordinates in the polar plane, and points in the Cartesian plane when given parametrically. Students should be able to graph basic polar functions, the primary technique being to first graph r vs θ in the Cartesian plane and then translating to polar coordinates by allowing θ to ‘travel’ from zero through one period of r. Students should also be able to graph basic parametric functions in the plane; the primary technique here is to find the relationship between the two coordinate functions and then allow one coordinate to ‘change’ to identify the direction and range the points will travel on the relationship function. (Part of the problem in this last case is to identify which coordinate function to investigate to determine the direction traveled.)

Technology – Graphing Calculators are of great help in graphing polar and parametric functions. Emphasis must be placed on the ‘window’ since the x-value for the window is no longer the independent variable, but is instead related to the ranges of the functions, as is y. For both polar and parametric equations, students must learn the role of ‘steps’ for both θ and t.
I. Consider the functions: $f(x) = x^2 + 3x + 1$ and $g(x) = 2x - 5$.
Find and simplify each expression.

(a) $f(x + h)$

(b) $f(g(x))$

II. Consider the polynomial $y = (2x - 1)^2(x + 4)^3$.

(a) Describe end behavior.

(b) Find the $x$-intercepts.

(c) Sketch an approximate graph using parts (a) and (b).

III. Solve for $x$: $\log_8(x - 1) - \log_8(x - 2) = \frac{1}{3}$
Rubric

I. (a) +1 understands function notation \( f(x + h) = (x + h)^2 + 3(x + h) + 1 \)
    +1 correct simplification
(b) +1 understands function composition \( f(g(x)) = (2x - 5)^2 + 3(2x - 5) + 1 \)
    +1 correct simplification

II. (a) +1 understands end behavior of polynomials
    (b) +1 understands x-intercepts
    (c) +1 behavior at x-intercepts (based on answer to part (b))
        +1 correct graph (based on parts (a) and (b), note: y-intercept not necessary)

III. +1 attempt to use log rules
    +1 correct use of log rules
    +1 understands definition of log (correct based on result of step 1)
    +1 correct solution (based on result of step 2)
I. A 30-foot cable is attached to the top of a pole and point on the ground. The cable makes a $50^\circ$ angle with the ground. Find the height of the pole.

II. Find all $x$ such that $2 \sin(2x) - \sqrt{3} = 0$
Rubric

I. +1 correctly interprets problem (with a picture)
   +1 sets up a correct equation using a trig function and height of pole
   +1 correctly solves for height of pole

II. +1 correctly solves for \( \sin(2x) \)
    +1 identifies angles that satisfy the equation (solves for \( 2x \))
    +1 correctly solves for \( x \)
    +1 proper usage of \( +k\pi \) to express all solutions
Math 205-206 (Calculus I-II) Course Outline  
UHH Mathematics Department  
2007

Catalog Description

Basic concepts of differentiation and integration with applications. Integrals of trigonometric, exponential and logarithmic functions; differential equations; techniques of integration and applications, infinite series. Pre: C or better in Math 104 or Math 104G, or Math Placement Score greater than 49 for enrollment in Math 205; C or better in Math 205, or Math Placement Score greater than 59 for enrollment in Math 206.

It should be noted that both Math 205 and 206 include 3-credit lecture and 1-credit computer-based lab components. The outline provided below is for the lecture portion of the course. The lab portion of the course is designed to enhance the concepts presented in the lecture by providing numerical representations of the concepts. The primary goals of the lab are to introduce the students to computer based algebra systems (CAS) and other available information technologies (e.g. spreadsheets, graphing calculators, etc.), to understand when the use of computer technology is appropriate and when it is not, to provide numerically based solutions to Calculus problems, and most importantly to enhance their understanding of the Calculus through computer based visualization capabilities. For certain topics (e.g. differential equations), the lecture instructor may choose for the topic to be covered entirely within the lab, but for most topics the intent is for the lab to proceed closely with the lecture, for the lab to enhance students’ understanding of the lecture material by presenting the same material from a numerical approximation perspective. The final grade for each course is weighted 75% lecture and 25% lab, with the lab instructor providing lab component feedback to the lecture instructor for determination of the final grade, which is determined by the lecture instructor.

Course Goals/Learning Outcomes Objectives for Math 205

The primary goals of Math 205 include providing students an introduction and then a strong background in the fundamental concepts and techniques associated with first semester Calculus, namely the Derivative and the Definite Integral. Math 205 is a prerequisite for Math 206, so an ancillary but important goal is to prepare students for success in Math 206. Care should be taken to ensure that students gain both a deep understanding of these concepts as well as becoming proficient in the associated techniques usually used to solve the Calculus problems. University mathematics typically places more emphasis on gaining a true understanding of concepts than students might have been exposed to in High School mathematics courses, and consequently an additional goal of Math 205 is increase students’ mathematical sophistication.

Students successfully completing the course should be able to:
- Compute limits both graphically and symbolically
- Determine if a function is continuous at each point
- Compute the derivative at a point using the definition
- Become proficient in using all rules of differentiation (see below) to compute
  global derivatives, and knowing when and when not to use each rule
- Apply the conceptual derivative to linear approximation
- Utilize the symbolic derivative to find the equation for a tangent line and use that
  line to [linearly] approximate functional values
- Explain in practical terms the concept of the derivative
- Utilize implicit differentiation to solve related rates problems
- Utilize the derivative to optimize functions
- Utilize the derivative and higher order derivatives to graph functions
- Understand and apply the derivative and higher order derivatives to modeling
  problems
- Utilize L’Hospital’s Rule and know how to deal with the different indeterminate
  forms
- Understand and apply the Fundamental Theorem of Calculus
- Use the definition of the Definite Integral to set up problem solving models
- Understand and apply Integration as a process (see note on the Definite Integral
  below)
- Anti-differentiate basic functions and utilize u-substitution where appropriate

**Topics Covered**
- Pre-Calculus Review (optional, but recommended)
- Limits and Continuity
- The Derivative
- Rules of Differentiation
- Applications of the Derivative
- The Definite Integral

**Pre-Calculus Review (1-2 weeks)**

See outline for Math 104, with particular emphasis on:

- Graphical Modeling and Interpretation
- Function Domains
- The fundamental properties of the exponential and log functions
- Unit Circle Trigonometry

**Limits and Continuity (2 weeks)**

Conceptual – instructors need to be careful to ensure that students truly understand the
concept of limit, as opposed to simply performing a lot of algebra and not being able to
interpret their results. This is not to say that students do not need to be able to perform
algebra, but algebra for the sake of algebra will only improve their algebraic skills.

- Graphical interpretation of limits (students should be able to identify limits, if
  they exist, or explain why they do not, without the use of formulas)
- Overall purpose of the concept of limit (e.g. $1 = \frac{9}{9}$, defining $\pi$, adding infinitely many numbers, instantaneous velocity and rate of change)
- Graphical left/right-hand limits
- Limits at infinity (utilize the concept of certain functions “growing” faster than others)
- Understand and apply the connection between limits and continuity

Skills
- Compute limits algebraically using basic algebraic techniques
- Compute limits at infinity using algebraic techniques
- Determine if piecewise defined functions are continuous using symbolic manipulation

The Derivative (3 weeks)

Conceptual
- Definition of the Derivative
- Instantaneous velocity and rate of change/slope of the tangent line
- Explain in practical terms the statement that $C'(100) = 10$

Skills
- Compute the Derivative of a function at a point, given symbolically, using the definition
- Find the equation for the tangent line to a function at a point
- Compute an approximation to the Derivative of a function given numerically, at a single point

Rules of Differentiation (3 weeks)

Skills
- Power Rule, Product Rule, Quotient Rule, Chain Rule, Exponential and Logarithmic Derivatives, Trigonometric Derivatives, Logarithmic Differentiation
- Computing higher order derivatives

Applications of the Derivative (3 weeks)

Conceptual
- Linear approximation from a conceptual view: Given $C(100) = 2000$ and $C'(100) = 10$, approximate $C(101)$.
- Related Rates: ideally, students can use the concept of the derivative to approximate solutions to related rates problems (optional)
- Apply the concept of the derivative and second derivative to sketch graphs
- Given the graph of the derivative of $f$, sketch a possible function $f$.
- Conceptual understanding of why L’Hospital’s Rule works
- Given graphs of $f$, $f'$, and $f''$ determine which is which
- Given a standard position function, utilize the derivative to understand movement
- Given the graph of the derivative of position, answer modeling questions regarding maximum speed, acceleration, and position

Skills

- Linear Approximation: given a symbolic formula for \( f \), approximate the value for \( f \) near a value for \( x \) at which \( f \)'s value and that of its derivative are easily computed symbolically (or given numerically via a table of values for example)
- Implicit Differentiation
- Related Rates – solve standard related rates problems
- Optimization – solve standard optimization problems
- Compute critical numbers from symbolically defined functions and use the symbolic derivative and second derivative to graph \( f \)
- Utilize L’Hospital’s Rule to compute indeterminate form limits

Definition of the Definite Integral (3 weeks)

Note: one of the primary goals of this portion of Math 205/206 is to get the students to truly understand the concept of “integration” as a process, the process of dividing things into small pieces, finding out information about each piece, and then summing up the results. It is only if they understand the “process” that they might stand a chance of tackling a non-traditional type of problem in the future.

Fundamental Theorem of Calculus

Conceptual
- Definition of the definite integral as a Riemann limit
- “area under” velocity = change in position, “area under” \( f' \) = change in \( f \)
- If you wish to find area under \( f \), find a function \( F \) such that \( F' = f \) and evaluate \( F(b) - F(a) \)
- If you wish to find change in \( f \), evaluate the definite integral for \( f' \) (either symbolically, graphically, or numerically, e.g. with your calculator). It is essential that the student knows when it is appropriate to use which
- Understand the fact that every continuous function over \([0,1]\), for example, has an antiderivative (namely the definite integral over \([0,x]\))

Skills (Anti-differentiation)

- \( \int_{a}^{b} f'(x) \, dx = f(b) - f(a) \) (Actually computing the value symbolically)
- Algebraic simplification (reducing to the sum of powers)
- Standard functions (polynomial, basic trig., basic exp., ln)
- U-Substitution
- Guess and Check (optional)
- Using the fact that every continuous function over \([0,1]\), for example, has an antiderivative (namely the definite integral over \([0,x]\)), use the chain rule to compute definite integrals of the form \([0,g(x)]\). (optional)

Numerical Integration (optional – generally covered in the lab)

Conceptual
- Approximating Definite Integrals using Left(n), Right(n), Trap(n), Mid(n), and Simp(n), and knowing under which conditions can you determine if each approximation is too large, too small, or indeterminable. (Note: Simp(n) can be represented as a weighted average between Trap and Mid…Simp(n) = (Trap(n) + 2Mid(n))/3)
- Given that f is monotonic, find the least n such that Left(n) is within $\varepsilon$ of $\int_{a}^{b} f$
- Conclusions that can be made regarding these approximations, given concavity (e.g. if f is monotonic and concave up over [a,b], then Trap(n) is too big, etc.)

Skills – covered in lab
Math 206 is a continuation of Math 205. It utilizes the derivative knowledge and introduction to the definite integral gained in Math 205 to build a strong understanding of integration, both as a process and a tool, and how to use integration to solve volume and other application problems. Once techniques of anti-differentiation are learned we then extend that knowledge to an introduction to differential equations and finally come full circle back to the derivative/approximations to Taylor Polynomials.

Students successfully completing the course should be able to:

- Understand and apply the Fundamental Theorem of Calculus
- Use the definition of the Definite Integral to set up problem solving models
- Understand and apply Integration as a process (see note on the Definite Integral below)
- Use the Definite Integral to find volumes by the washer/shell/slicing methods
- Use the Definite Integral to solve work problems
- Identify appropriate methods of anti-differentiation (i.e. u-sub, parts, trig, trig sub) and be able to apply the methods
- Solve differential equations from slope fields, identify equilibrium solutions and stability, apply Euler’s Method, separation of variables, and by multiplying through by an integrating factor (optional, covered in the lab)
- Identify whether or not a series converges utilizing various tests (e.g. terms must go to zero, alternating series, absolute convergence, root/ratio test, comparison)
- Identify certain types of series (e.g. geometric, harmonic) and be able to compute the values for geometric series
- Compute radius and interval of convergence for power series
- Compute [at least] the first few values for a Taylor Polynomial
- Compute and manipulate McLauren Series for sin, cos, exp

Note: There is the definite possibility of overlap between the material at the end of Math 205 and the beginning of Math 206. The amount of overlap beginning in Math 206 is entirely up to the instructor. While minimal review of the definition of the Definite Integral is certainly warranted in all cases, some instructors may choose to cover the subject again in great detail, which is fine provided adequate time is left for subsequent topics.

Topics Covered
- Definition of the Definite Integral
- Applications of the Definite Integral
- Antidifferentiation Techniques
- Differential Equations
- Infinite Series
The times allotted for each subject are of course only suggestions and as such should be used only as guidelines. Some instructors do not cover much on the definition and spend more time on applications and anti-differentiation, and some instructors skip differential equations in the lecture portion of the class. Care should be taken to allow enough time for covering Taylor Series in the infinite series portion of the course.

**Definition of the Definite Integral (3 weeks)**

Note: one of the primary goals of this portion of Math 205/206 is to get the students to truly understand the concept of “integration” as a process, the process of dividing things into small pieces, finding out information about each piece, and then summing up the results. It is only if they understand the “process” that they might stand a chance of tackling a non-traditional type of problem in the future.

**Fundamental Theorem of Calculus**

Conceptual
- Definition of the definite integral as a Riemann limit
- “area under” velocity = change in position, “area under” \( f' = \) change in \( f \)
- If you wish to find area under \( f \), find a function \( F \) such that \( F' = f \) and evaluate \( F(b) - F(a) \)
- If you wish to find change in \( f \), evaluate the definite integral for \( f' \) (either symbolically, graphically, or numerically, e.g. with your calculator). It is essential that the student knows when it is appropriate to use which
- Understand the fact that every continuous function over \([0,1]\), for example, has an antiderivative (namely the definite integral over \([0,x]\))

**Skills (Anti-differentiation)**
- \( \int_{a}^{b} f' = f(b) - f(a) \) (Actually computing the value symbolically)
- Algebraic simplification (reducing to the sum of powers)
- Standard functions (polynomial, basic trig., basic exp., ln)
- U-Substitution
- Guess and Check (optional)
- Using the fact that every continuous function over \([0,1]\), for example, has an antiderivative (namely the definite integral over \([0,x]\)), use the chain rule to compute definite integrals of the form \([0,g(x)]\). (optional)

**Numerical Integration (optional – generally covered in the lab)**

Conceptual
- Approximating Definite Integrals using Left(n), Right(n), Trap(n), Mid(n), and Simp(n), and knowing under which conditions can you determine if each approximation is too large, too small, or indeterminable. (Note: Simp(n) can be represented as a weighted average between Trap and Mid…Simp(n) = \((\text{Trap}(n) + 2\text{Mid}(n))/3\))
- Given that \( f \) is monotonic, find the least \( n \) such that Left(n) is within \( \epsilon \) of \( \int_{a}^{b} f \)
Conclusions that can be made regarding these approximations, given concavity (e.g. if $f$ is monotonic and concave up over $[a,b]$, then $\text{Trap}(n)$ is too big, etc.)

Skills – covered in lab

Applications of the Definite Integral (3 weeks)

Note: emphasis here should be on setting up the definite integrals...again, towards an understanding of the integration process. Computing the values is covered in the Definite Integral above. A useful technique in this regards might be to emphasize that setting up the Riemann limit, which then according to the definition yields an integral, requires nothing more than identifying the value for one single washer/shell/slice/arc-length/work/pressure, etc.

Conceptual
- Volumes of Revolution (washer and shell methods)
- Volumes by slicing
- Arc Length (one might want to emphasize here that despite the fact that the “obvious” method of finding arc length does not yield a definite integral according to the Riemann definition above without first applying an algebraic trick along with continuity of the first derivative, utilizing a computer to get an approximation would usually yield the result and hence is a completely reasonable approach)
- Work (primarily, work = weight x distance moved)
- Optional (Pressure, Center of Mass, Probability, etc.)

Skills
- Visualizing solids of revolution, 3-dimensional objects in general, work, etc.
- Finding points of intersection and determining limits of integration
- Review of simple anti-derivatives and computing the definite integral (Optional – since the emphasis here is setting up the integrals, computing them can be done via technology if the instructor wishes)

Antidifferentiation Techniques (3+ weeks)

Emphasis here is to learn the basic techniques, and is therefore primarily skill oriented. Many instructors find it useful to assign take-home work in order to allow the students the necessary time to become proficient without spending too much time in class. Again, this is entirely at the discretion of the instructor.

- u-sub
- parts
  - standard parts (including tabular parts)
  - circular parts
  - parts applied to ln, inverse trig, etc.
- basic trigonometric anti-differentiation (e.g. $\sin^3 x \cos x$, $\sin^3 x$, $\tan x \sec^2 x$, etc.)
- trigonometric substitutions (x-sub)
 improper integrals (some instructors prefer to cover this section when they
cover infinite series)
- partial fractions (optional, covered in the lab)

Differential Equations (2 weeks)
Note: this topic is optional in the sense that some instructors prefer to cover the material
in the lab. When covered in class it is mostly conceptual, with a little skill intermixed.
Conceptual
- Slope fields (both drawing and utilizing to approximate solutions)
- Equilibrium solutions for \( y' = f(x, y) \) (Set \( y' = 0 \), check if you get a constant
solution (i.e. \( y = c \))
  - Stability (check visually from slope field, and numerically by asking if \( y < c \), is \( y' > 0 \), causing \( y \) to return to \( c \), etc.)

Skills
- Determine whether \( f \) is a solution to a differential equation (simply take the
necessary derivatives and “plug them into the appropriate slots”)
- Euler’s Method
- Solving separable equations
- Exact equations (Optional)

Infinite Series (5 weeks)
Note: The ultimate goal here is to gain an understanding of power series and more
importantly Taylor Polynomials. All too often time runs out and the students only think
of infinite series in terms of the convergence rules and techniques, with emphasis only on
specific series and not series-defined functions. So, care must be given to allow enough
time.

The study of infinite series intermingles the conceptual and the necessary skills
throughout. What follows is a useful outline on the subject.

General Convergence of \( \sum a_n \):
- Is \( \lim_{n \to \infty} a_n = 0 \)? (Necessary, but not sufficient. In other words, if yes, it might
  converge; if no, then it definitely does not. You should always check this
  FIRST.)
- Is it an alternating series? (If yes, then combined with a above, it converges.)

Recognizing Series:
- Harmonic Series \( \sum \frac{1}{n} \) diverges.
- Geometric Series \( \sum r^n \) (converges if \( |r| < 1 \), diverges otherwise)
- P-series \( \sum \frac{1}{n^p} \) (converges if \( |p| > 1 \), diverges otherwise)

Computing the value of series:
Generally this is not possible. But, for certain types of series, like telescoping (optional) and geometric, it is possible. You should know the trick for finding the partial sum for geometric series, and then for computing the value (of course the value is always $\frac{1}{1-r}$ if the series begins with a 1).

Tests for convergence (not for finding the sum, just whether it exists or not)

- Comparison (are the terms of the series less than a series that you already know converges, such as a geometric or p-series? If yes, then the series converges. Are the terms of the series greater than the terms of a divergent series such as the harmonic, or a geometric or p-series that diverges? If yes, then the series diverges.)
- Integral Test (optional)
- Root/Ratio tests (some students benefit from seeing that the reason these seemingly abstract tests work is that they easily compare to geometric series)

Power Series

- Find the radius of convergence (usually this entails finding the values of x that would work for the “r” in the geometric series, p-series, or root/ratio tests).
- Find the interval of convergence (just check the endpoints from j above)

Taylor/Maclaurin Series

- Know the purpose of investigating Taylor Polynomials (Here students should understand that the easiest types of functions for us to investigate are polynomials, as they are algebraic and easily computed. The next best thing for a non-polynomial, such as sin or exp is to be able to represent them as “infinite polynomials” which we refer to as power series or when applied to a given known function such as sin, as Taylor Polynomials)
- Know the general form for the Taylor Polynomial
- Know the Taylor Polynomials for the exponential, sin, cos.
- Be able to determine how many terms are necessary (i.e. which “finite” polynomial can be used to) to approximate sin (close to zero for example) to within a desired pre-determined error.
- Be able to compute the Taylor Polynomial for general functions (or at least the first few coefficients).
- Check if a series is a Taylor Polynomial for a given function. (Note: students should know that the Taylor Polynomial for a convergent power series is itself, and as such each convergent power series is a Taylor Polynomial. But they should be able to compute the Taylor Polynomial for other functions as well.)
- Differentiate or integrate a Taylor Polynomial that is known, to get other Taylor Polynomials that would be difficult to compute term by term.
1. Evaluate, but do not simplify. \( \frac{d}{dx} [(3x^2 + 1) \cos (e^x)] \)

2. Evaluate, but do not simplify. \( \frac{d}{dx} \left[ \frac{4 \ln(x) + \tan(x)}{xe^x} \right] \)

3. Evaluate \( \int \frac{1}{\sqrt{x}} \sin (4\sqrt{x}) \, dx \)
Problem 1

1. Identify Product
2. Identify Power rule for \((x^2 + 1)\)
3. Identify Chain rule for \(\cos(e^x)\)
4. Chain Rule Applied Correctly
5. Correct Derivative

Problem 2

1. Identify Quotient
2. Apply Quotient Rule correctly
3. Identify denominator as product
4. Apply product rule
5. Correct Derivative

Problem 3

1. Identify as u-substitution and make some substitution
2. Make Correct Substation
3. Substitute and find \(du\)
4. Evaluate (attempt) new antiderivative
5. Correct antiderivative, with arbitrary constant
1. $D_x (3x^2 + 1)\cos(e^x) = _________________________________$

2. $D_x \frac{4\ln x + \tan x}{xe^x} = _________________________________$

3. $\int \frac{1}{\sqrt{x}} \sin(4\sqrt{x}) \, dx = _________________________________$
1. \( D_x (5x^{10} - 12x^3 + \pi^2) = \) 
   
2. \( D_x x^2 \cos x = \) 
   
3. \( D_x \left( \frac{e^x + x^e}{\sqrt[3]{x}} \right) = \) 
   
4. \( D_x (e^x \sin x)^6 = \) 
   
5. \( D_x \sin^2(3x^4 + 5x^2 - x) = \) 
   
6. \( D_x \ln(\sec \sqrt{x^2 - 1}) = \) 
   
7. \( D_x \sqrt{\tan(e^x)} = \)
Differentiation – We may want to use a gateway exam for this subject, with a mandatory passing grade of 80% (no algebraic simplification, just applying the differentiation rules). The following is a Gateway Exam used in the past to test differentiation, the rules and the basic idea. The base-2 exponentials could be changed to the natural base, and of course other changes could be suggested as well.

1. \( D_x \sin 3x = \) __________________________

2. \( D_x e^{-2x} = \) __________________________

3. \( D_x x^2 2^x = \) __________________________

4. \( D_x 4x^5 + \frac{2}{x} + e^2 = \) __________________________

5. \( D_x x^2 2x^2 = \) __________________________

6. \( D_x \frac{\tan \sqrt{x}}{e^{\sin x}} = \) __________________________

7. \( D_x \ln(x^2 \cos^2 x) = \) __________________________

8. \( D_x \sin^2 x^2 = \) __________________________

9. \( D_x (3x^2 + 2x)^3 = \) __________________________

10. The odometer reading \( O(t) \) (in miles) on a car at time \( t \) seconds is given by the following table. Approximately how fast (in miles/sec) was the car traveling when it had traveled 42,246 miles?

<p>| | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
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<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>t</td>
<td>0</td>
<td>30</td>
<td>60</td>
<td>90</td>
<td>120</td>
<td>150</td>
</tr>
<tr>
<td>O(t)</td>
<td>42,244</td>
<td>42,244.5</td>
<td>42,245</td>
<td>42,245.7</td>
<td>42,246.1</td>
<td>42,246.4</td>
</tr>
</tbody>
</table>
Linearization

Choose one of the following:

1a) Suppose the number of gallons of pollution, $P$, in a lake on the 150th day of observation is 100,000 (i.e. $P(150) = 100,000$ gallons) and $P'(150) = 10$. Without any further information, what is your best approximation of the amount of pollution on day 153?

______________________

b) Use linear approximation to approximate $\sqrt{1000.00000000000000000000000000006}$

(Note: $1000.00000000000000000000000000006$ is the same as $1000 + 6 \times 10^{-25}$.) (note, this can be stated without the use of all those zeros, but the number of zeros should be of such a magnitude that a calculator will not give an answer closer than 10.)

Or, it can be something like, given $f(1) = 10$ and $f'(1) = 2$, approximate $f(1 + 3\times10^6)$; or find the equation to the tangent line to a function at a point, and then use that to approximate a value close by.

Optimization

Here, a basic optimization problem should suffice, such as, find the dimensions of a window in the shape of a rectangle with an equilateral triangle on top (give sketch) such that the outside perimeter of window is 12 feet and the amount of light passing through the rectangular portion of the window is maximized.

Or, something a little easier, such as, A dog trainer has 200 ft of fence with which to make kennels for 5 dogs. The kennels will be constructed side-by-side in order to ‘share’ the fence material between kennels. (See the sketch below.) What dimensions should the trainer use for the individual kennels in order to provide each dog with the most possible area in which to move around, and what will be the resulting area? (Note: assume the fence can be cut and pieced together as you please.)
Graphing

1. Graph a possible function $F$ below, which is defined for all real numbers and has the following properties.

<table>
<thead>
<tr>
<th></th>
<th>$(-∞, -4)$</th>
<th>$(-4, -1)$</th>
<th>$(-1, 2)$</th>
<th>$(2, 3)$</th>
<th>$(3, 5)$</th>
<th>$(5, ∞)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F'$</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>$F''$</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>+</td>
</tr>
</tbody>
</table>

A more sophisticated version of this is to give a graph of $F'$, with units included for $x$, and ask to draw a possible continuous function $F$.

A more manipulation driven example is something like Given $f(x) = 2x^3 + 3x^2 – 12x – 5$, find the interval(s) where $f$ is increasing. Show all work.

Antidifferentiation and the Fundamental Theorem of Calculus

1. \[ \int x \cos(x^2) dx = \]

2. \[ \int_1^2 x^3 + 3x - 4 \, dx = \]

3. There seems to be something going on at the UHH cafeteria. The graph below depicts the rate at which people are arriving (or leaving) the cafeteria between noon ($t = 0$) and five minutes past noon ($t = 5$).

The number of people arriving at the cafeteria was exactly the same as the number leaving the cafeteria at time $t = \text{__________}$. The cafeteria was the fullest at time $t = \text{__________}$. If the cafeteria had 100 people in it at noon, how many were there 5 minutes later? \text{__________}
These writing assignments are intended to improve your understanding of key concepts and to better prepare you for the larger Group Project writing assignments. It is intended that these smaller writing assignments should be no more than a page in length, and in many instances students will find that they can do a sufficient job with two paragraphs. However, it is imperative that you follow the guidelines I provide herein.

One of the more difficult challenges you will face in your education and professional careers is to succinctly put into writing your valid and useful thoughts, with just enough detail to elucidate the reader without requiring them to do much thinking. You will find this is what most readers want, and many supervisors require. With respect to mathematics, most do not want to know the details. Yet, they all want to be assured that the procedure is valid, and that if they viewed the details they would be consistent with their limited understanding. It is your responsibility to provide such an understanding without too many details.

Please follow the guidelines below for this writing assignment. You are not required to use subheadings in your paper.

- **Problem Statement**: what is it you are trying to find or solve?

- **Formula(s) you wish to use**: what formulas or tools do you have at your disposal or that you are using or attempting to use? For example, the distance between two points formula, distance traveled formula, volume of a box or cylinder, etc.

- **Obstacle(s)**: why is the problem difficult? In my opinion this is the most important aspect of your exposition. Once you’ve pointed out to the reader the difficulty, you can then orient the reader towards your strategy for obtaining a solution.

- **Strategy**: what is your strategy for overcoming the difficulty mentioned above? And, why does your strategy overcome that difficulty?

- **Justification**: why should I be confident that your strategy, and hence your solution, is valid? Can you give me some kind of indication that your results make sense?

**Assignment**: Use the template above to answer the following problem.

A train is travelling along a straight and level path in such a way that its position at time $t$ (in hours past noon) is given by $P(t)$. Here $P(t) = 0$ means the train is at the train station at time $t$, and $P(t)$ (in miles) is positive if the train is north of the station. Explain how to find the train’s instantaneous velocity at time $t = 2$. 
Dear Calculus Researchers,

I co-host a popular television show called Mythcrushers on the Integration Cable Network. We are requesting your help in “crushing” one of our myths. Have you ever heard an idea described as a “lead balloon?” Well, the whole basis of that saying is the thought that you just can’t make a lead balloon take off. We are going to attempt to build our very own lead balloon and put to rest the “lead balloon” description forever! Unfortunately, after our initial work, we are running into some problems figuring out how big our balloon needs to be. Let me fill you in with our plans. Our balloon shape, for a given height $L$, will be precisely given by revolving the function $u(x; L)$ for $0 < x < L$ about the $x$-axis. (For example $u(x; 10)$, $0 < x < 10$ would give the profile for a 10 foot high balloon.) The function $u$ is given by

$$u(x; L) = \frac{L}{4} \left( 16.1 - \left( \frac{4x}{L} \right)^2 \right)^{\frac{3}{2}} - \frac{4L}{7} e^{-\left(\frac{2x}{L}\right)^2}.$$

What we need to know is how tall, or what value of $L$ in feet, do we need to make the balloon in order to lift itself (a lead skinned balloon is hefty!), a 250lbs basket and two 175lbs passengers. We will fill the balloon with helium gas which can lift 1lbs for every $16.1 ft^3$ of gas. The amount of helium that an $L$ foot high balloon can hold can be found by finding the volume of the solid formed by revolving $u(x, L)$ about the $x$-axis from 0 to $L$. In order to find the weight of all the lead foil that we will use to build the skin of the balloon, find the surface area of the balloon and use the fact that our foil is $4.16 \times 10^{-5} ft$ thick and lead weighs $708 lbs/ft^3$.

We would like you to find the smallest height $L$, to the nearest foot (but better would be appreciated!), that makes our balloon float. Since our design is complicated, you will need to solve this problem numerically on a computer.

To keep with our television schedule and to give us sufficient build time we request your typed final report, including supporting calculations, by Friday, February 17th. So we don’t spoil the surprise of this project, we ask you to speak only to Professor Wissman and lab Instructor Mr. Garry. We have discussed this project with Professor Wissman and he is prepared to help you with any questions you may have. Thank you for your help and expertise! We look forward to your reply.

Sincerely,

Adam Vicious
Project Checklist

Dr. Brian Wissman

- This list will be used to grade your assignment and will be returned to you with comments. Keep a copy of your paper for your own reference.
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Comments:
Math 206 Assessment Problems

1. \[ \int_{1}^{2} \left( x^{3} + 3x - 4 \right) dx = \]

Award one point for each of the following:

- antidiff (even if incorrect)
- \( F(b) - F(a) \) with correct order and subtraction
- correct numerical answer (even if only by calculator)

2. There seems to be something going on at the UHH cafeteria. The graph below depicts the rate at which people are arriving (or leaving) the cafeteria between noon (t = 0) and five minutes past noon (t = 5).

The number of people arriving at the cafeteria was exactly the same as the number leaving the cafeteria at time \( t = \) ___________. The cafeteria was the fullest at time \( t = \) ___________. If the cafeteria had 100 people in it at noon, how many were there 5 minutes later? ___________

Award one point for each of the following:

- Correct answer for first blank
- Correct answer for second blank
- Recognizing that it is a FTC problem and calculate an area (even if incorrect)
- Adding the area to 100 for the last blank (even if use wrong area)
- Everything correct
3. Set up volume of revolution about the x axis.

Award one point for each of the following:

- choosing the washer method
- writing out the correct limit sum with correct use of $w_i$
- write out definite integral with correct limits of integration (even if they use an incorrect answer from the second bullet
- correct numerical answer based on bullet 3 (use algebra or calculator)

4. Antidifferentiation techniques. Choose basic parts, u-sub, and trig sub.

a. $\int \frac{x}{\sin x^2} dx = \quad \text{Award one point for each of the following.}$

- Identify problem as a u-substitution
- Correct Substitution chosen
- Find $du$ from chosen $u$ (even if $u$ is not the correct choice.)
- Give equivalent integral in terms of the new variable with their chosen $u$
- Correct answer

b. $\int x^2 \sin 3x dx = \quad \text{Award one point for each of the following.}$

- Identify integration by parts as the correct technique
- Choose correct $u$ and $dv$
- Identify that you must integrate by parts twice, or use tabular integration by parts.
- Divide by 3 when anti-differentiating
- Correct answer

c. $\int \frac{1}{\sqrt{x^2 - 1}} dx = \quad \text{Award one point for each of the following.}$

- Identify problem as a trigonometric substitution
- Correct substitution chosen
- Find $dx$ (even if differentiation is incorrect) and replace $x$ with their chosen substitution. (Even if substitution chosen is incorrect.)
- Evaluate
- Back substitute correctly for $x$ with their chosen substitution
5.  Differential Equations – is \( y \) a solution to a differential equation. Is the equilibrium
stable or unstable, and why. Is \( y = \frac{-1}{x + 2} \) a solution to the differential equation \( \frac{dy}{dx} = y^2 + 2 \)? (Please show your work.)

Award one point for each of the following:

- Differentiate \( y \)
- Differentiate correctly
- Plug in for \( y \) and \( dy \) (even if \( y' \) was incorrect)
- Simplify equation correctly
- Determine if it's a solution or not based upon their work

6.  Find the interval of convergence of the power series \( f(x) = \sum_{n=1}^{\infty} \frac{x^{2n}}{n} \)

Award one point for each of the following:

- Identify the need for the Ratio or Root test to test for convergence
- Give an open interval of convergence, even if wrong.
- Correctly identify convergence for \(-1 < x < 1\).
- Identify that the series diverges at \( x = -1 \) (must show work)
- Identify that the series diverges at \( x = 1 \) (must show work)

7.  Find the Taylor polynomial centered at \( x = 0 \) for \( 2x^2 e^{-x} \).

Award one point for each of the following:

- Substitute \(-x\) into the power series for \( e^x \) or just give the power series for \( e^{-x} \)
- Correctly add two powers of \( x \) to each term in the series, even if the above series is incorrect
- Multiply power series by 2, even if the wrong series
- Correct Answer

8.  Find the Taylor Series centered at zero for \( f(x) = e^{-x^2} \), and use the definition from 5a above to verify the correctness of your first two non-zero terms. (Show Work)

Award one point for each of the following:

- Substitute \( x^2 \) into the power series for \( e^x \), even if incorrect
- Identify that the value is correct for \( x = 0 \)
- Identify the need to differentiate the function twice
- Check that the coefficients match (even if incorrect)
- Correct answer (i.e. coefficients match perfectly)
The above schematic outlines Greene Hilly estate. The Raging River follows $y = \sin x$, the boundary is a circle of radius 3, the driveway runs from the West Gate to the East Gate along the parabola $y = x^2$, and the mansion is located at the point 1.5 miles east, 2 miles north of the south bridge. (Note: although the bridges are not marked, there are clearly two, each located at a point where the driveway must cross the river.)

It is rumored that Lord Greene made his fortune in some unscrupulous manner, but he is semi-retired now and devotes much of his time to his grandson, Little Johnny. (What do we care, as long as he pays us well?) Lord Greene seems to be satisfied with our work here at CompuCalc and has again hired us as consultants for work related to his advanced technology company, FutureFun. This time he wants us to help Little Johnny with his new toy, the River Skipper. According to my intelligence, the River Skipper is a small boat (about 3 ft in length) that has been programmed at the factory, utilizing GPS technology, so that it will always travel down the middle of the Raging River, and so that its speed at any point in time is determined by its location on the river. The coordinate system it uses for the Raging River is determined by setting the origin to be the center of the estate, with the x and y coordinates respectively representing...
miles east and north of center. The only requirement from Little Johnny is that he releases the boat at some point on the Raging River. The Skipper then moves along in the middle of the river so that its speed at each point (x,y) on the river is given by a velocity function v, which depends only on the x-coordinate, and the direction it travels along the river is such that if v > 0 then it moves so that its distance east of the center (i.e. its x-coordinate) increases and if v < 0 it moves so that its distance east from center decreases. The engineers at FutureFun have also informed Lord Greene that they currently use the setting $v_1(x) = 1 + 5 \cos^2 x$ for the velocity, where the units for v(x) are miles/hour, but are considering using $v_2(x) = 5 \cos^2 x^2$.

Since Little Johnny likes to play around the south bridge (the center of the estate), Lord Greene intends to suggest to Little Johnny that he put the boat in the river there, and then run (in a straight line) to the north bridge to retrieve the boat. The concern that Lord Greene has is whether or not Johnny can run (or is it walk?) fast enough to get there before the boat arrives. Note: the technology is sophisticated enough so that you can ignore any current in the river. He also needs to know the advantages and/or disadvantages of using either of the two available velocity settings.

Please write a one-two page report to me, the CEO of CompuCalc, explaining how fast Little Johnny will need to travel to retrieve the River Skipper at the North Bridge, for each of the velocity settings. Also, just out of curiosity, I would like to know how fast little Johnny would have to run if he follows the driveway as opposed to running to the north bridge in a straight line, as running along the driveway is probably much easier than through the woods.
Group Project Template

Your group project reports should follow the template below.

**Introduction**
In this section you should include a reader friendly introduction to the problem, so that an outsider can understand what the project is all about. Details should be left to the next section.

**Problem Statement**
In this section you should include details that are given to help frame the problem.

**Objectives**
In this section you should include the primary objective(s) of the problem. Usually there is one primary objective, which should be clearly stated…what it is the client wants answered. In many instances answering the primary objective reduces down to finding one particularly difficult secondary objective. At the very end of this section, so that it transitions well to the next, you should clearly state WHY this secondary (or primary if there is only one) result is difficult. Usually this is because you cannot use a particular standard formula or technique. You should state this clearly. This is one of the most important ingredients of your report, since the rest will flow from here.

**Strategy**
Here is where you want to state, in clear everyday terms, the strategy you used to overcome the difficulty mentioned above, and why this strategy addresses that difficulty. If you are using approximations, your strategy should also include the method you intend to use to ensure your approximation is valid. In other words, it is up to you to convince your supervisor that your results are reasonable and reliable. You do NOT need to include minor step in your processes, those will be included in the next section.

**Methodology**
Here is where you want to explain the steps you followed to apply the pieces of your strategy. If you use standard formulas you do not need to show the details of how you applied them, you can simply state that you applied them. But, make sure the reader can follow how you applied them. For example, if you use the distance between two points formula, you need to make it clear to
which two points you are referring. This section should support your strategy in such a way that
the reader can easily follow the steps since you already explained the strategy.

**Results**

Here is where you want to summarize the results you obtained from the methodology you
applied. It is often the case that there will be multiple results about multiple objectives. Even
though the secondary objective was not the main objective, results for the primary difficulty
portion of your work, which should be the hardest from a mathematical perspective, should be
included along with a summary of the results of the main objective. The client will want the
main objective results, but your supervisor will want other results to gauge the reliability of your
work.

**Conclusion**

This should be a short paragraph that again summarizes the result and anything else you think
should be included to tie off loose ends.
### Writing Assignment Rubric

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### Writing Assignment Rubric 2

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<td>Problem Statement 2</td>
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<tr>
<td>Identify why FTC Fails</td>
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<td>What would the integral represent?</td>
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Math 206 Writing Assignment #1

Assumptions for this problem.

1. F is a continuous function that is positive valued between 1 and 3.
2. F is not given by a nice formula. It is, as all functions are, a collection of points.
3. You are not allowed to simply state “Use Math-9 on your Calculator.”

Explain how to find the area bounded above by F and below by the x-axis, to the left by the line x = 1 and to the right by the line x = 3. (Please follow the writing assignment template as close as possible.)

Math 206 Writing Assignment #2

The Fundamental Theorem of Calculus implies that if an object moves in one direction along a straight and level path in such a way that its speed at time t is given by a function of time, say v(t), then the “area” under v from time a to time b will represent how far the object moved during the time interval [a, b].

1. Suppose a train is travelling along a straight and level path running north and south in such a way that its velocity v(t) (in miles per hour) at time t (in hours past noon) is given by $v(t) = 20t\ln t$. (Note: this equation is closely related to the one used in Def Int HW #1.) Explain why $\int_{1}^{3} 20x\ln(x) \, dx$ represents how far the train traveled from 1:00 to 3:00. In other words, explain why the area under this velocity function from t = 1 to t = 3 gives the distance traveled. (In effect I am asking you to explain why the Fundamental Theorem of Calculus as stated above works.)

2. In some cases the speed of an object moving along a straight and level path may be determined by its position on the path instead of time, such as is the case with the River Skipper (Group Project 1). Explain why the Fundamental Theorem of Calculus as stated above fails for the River Skipper. What would $\int_{a}^{b} v(x) \, dx$ represent?

Please follow the writing assignment template as close as possible.
Writing Assignment #3

Assume \( f \) is a real valued function defined for all positive \( x \). Explain how to determine if the “area” under \( f \) exists from 1 to \( \infty \).

Please follow the writing assignment template as close as possible. You may assume you already know how to compute Definite Integrals.

Writing Assignment #4

Explain how to use Euler’s Method to find \( f(2) \), given \( f(1) = 3 \) and \( f'(x, y) = x + xy \).

Please use the writing assignment template.
These writing assignments are intended to improve your understanding of key concepts and to better prepare you for the larger Group Project writing assignments. It is intended that these smaller writing assignments should be no more than a page in length, and in many instances students will find that they can do a sufficient job with two paragraphs. However, it is imperative that you follow the guidelines I provide herein.

One of the more difficult challenges you will face in your education and professional careers is to succinctly put into writing your valid and useful thoughts, with just enough detail to elucidate the reader without requiring them to do much thinking. You will find this is what most readers want, and many supervisors require. With respect to mathematics, most do not want to know the details. Yet, they all want to be assured that the procedure is valid, and that if they viewed the details they would be consistent with their limited understanding. It is your responsibility to provide such an understanding without many details.

You should follow the outline below for all writing assignments in this class.

- Problem Statement: what is it you are trying to find or solve?
- What formulas or tools do you have at your disposal or that you are using or would like to use? For example, the distance between two points formula, distance traveled formula, volume of a box or cylinder, etc.
- Why is the problem difficult? In my opinion this is the most important aspect of your exposition. Usually this involves the desire to use the formula above, but for some reason you cannot in this setting. Once you’ve pointed out to the reader the difficulty, you can then orient the reader towards your strategy for obtaining a solution.
- What is your strategy for overcoming the difficulty mentioned above? And, why does your strategy overcome that difficulty?
- Why should I be confident that your strategy, and hence your solution, is valid? Can you give me some kind of indication that your results make sense?
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| Comments: | |
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Math 311 Course Outline
UHH Mathematics Department

Course Description


Prerequisites

Math 206 or consent of instructor. (Math 310 or CS 215 is also desirable.)

Course Goals

The primary goal is to introduce the students to the rigorous theorem-based structure of advanced mathematics. The secondary goal is for students to become familiar enough with the particular structure of linear spaces so as to be able to recognize non-standard linear spaces such as C[0,1] or the solution space for ordinary homogenous linear differential equations, and to use general linear algebra knowledge gained in this course to investigate these spaces.

Use of Technology

Given the primary goals of the course above, the role of technology should be to simplify numerical calculations (e.g. computing determinants, matrix inverses, or Eigenvalues/Eigenvectors) for either student work or instructional purposes.

Primary Topics

Systems of Linear Equations
Determinants
Vector Spaces
Linear Transformations
Eigenvalues and Eigenvectors

The latter three topics are by far the most important; but they require the ability to solve linear systems, and an understanding of the implications of the determinant to such solutions. Given time constraints, therefore, every effort should be made to quickly proceed through the first two areas of study.

Optional Topics

Graham-Schmidt Orthogonalization
Inner Products
Course Objectives

Systems of Linear Equations (2 weeks)

The primary objective of this section is to provide the student with enough background to be able to determine linear dependency and to compute the rank and nullity of Linear Transformations later in the course. The work is comprised of two distinct but complimentary areas, the mechanics of using matrices to solve linear systems and gaining an introductory understanding of the structure of such solutions. The areas can be represented as follows:

- Matrix representation of linear systems
- Basic row reduction
  - The mechanics of Gauss-Jordan row reduction
  - How to represent infinite solutions
- The use of matrix inverses to solve systems with unique solutions
- Theorems regarding the structure of the solution space with an eye towards later sections regarding rank and kernel of linear transformations, including but not limited to:
  - The main structure theorem...the solutions to non-homogeneous systems are of the form A + B where A is a single solution to the non-homogeneous system and B is the [possibly trivial] space of solutions to the associated homogeneous system.

Determinants (2 weeks)

The primary objectives of this section are to learn to compute determinants, the algebra of determinants (e.g. \(\text{Det}(AB) = \text{Det}(A) \text{Det}(B)\)), and the implications of the determinant to solutions of systems of linear equations (i.e. non-zero determinant implies unique solution, zero determinant implies either no solution or infinite solutions). Ample time generally does not exist to cover all the many properties of determinants, or even Cramer’s rule, which is a favorite of Engineers, but not necessarily mathematicians. Technology can be very useful when computing determinants, either in class or on assigned homework.

- Co-factor expansion (since it generalizes past the 3x3)
- The relationship between the determinant and the existence of inverses, and the structure of the solution to linear systems

Note: The determinant is needed later to formulate the characteristic equation in order to compute Eigenvalues.

Vector Spaces (5 weeks)

This chapter represents the students’ first attempt at independent proof. \(\mathbb{R}^n\) should be introduced as a concrete example, but other algebraic definitions for ‘plus’ and ‘scalar
multiplication’ should be emphasized. Matrices form an elementary starting point for this, but more theoretical spaces such as function spaces should be emphasized. Care should be taken to identify the ‘large’ spaces as vector spaces, such as the function spaces from $R^m$ to $R^n$ under the usual function operations, or the space of $n \times m$ matrices under the usual operation. This allows the amount of work the students need to show when demonstrating, for example, that the set of all $3 \times 3$ diagonal matrices form a vector space, or the set of real valued differentiable functions form a vector space. Once the concept of vector space/subspace is under control, attention should shift to bases, which requires a good understanding of linear independence/dependence.

- Determining whether an ordered Triple $(S, \otimes, \otimes)$ is a vector space
  - Use a variety of examples, both standard and non-standard (particularly with respect to the operations)
- Subspace theorem
  - Emphasize the need to only show closure of the two operations
  - Ensure students know the difference between subspace and subset
- Linear Combinations
- Span
  - Associated Theorems (e.g. equivalent cardinality of two linear independent spanning sets, what if a set spans a space – must it be linearly independent, etc.)
- Linear dependency and bases
  - Use matrices and determinants to determine if sets of vectors are linearly independent
  - Identify canonical bases for non-standard spaces such as $P_2$.

**Linear Transformations (4 weeks)**

The purpose of this chapter is to gain an understanding of linear transformations as linear functions operating on spaces such as $R^n$. Ultimately, the goal is to use eigenvectors and Eigenvalues to help in identifying the effects on an object operated on by such a function. Thus, this chapter should be spent familiarizing the student with how to identify when a function is linear or not, the implications to computing values once it is determined that the transformation is linear (i.e. the image of a basis determines the entire image), and the use of the Ker and Rank to identify the dimension of the image of a transformation. Examples should include important operators such as the derivative.

- Identify whether or not $T$ is linear.
- Understand the implications of $T$ being linear.
  - Compute $T(X)$, given $T(B)$, where $B$ is a [not necessarily canonical] basis for $\text{Dom}(T)$.
  - Describe geometrically $T(A)$, where the boundary of $A$ is piecewise linear in $R^2$.
- Compute $\text{Ker}(T)$, $\text{Nullity}(T)$, and $\text{Rank}(T)$
  - Note: in many instances this involves solving homogeneous linear systems, again back to section one.
- Describe geometrically $\text{Ker}(T)$
- Compute matrix representation for $T$

**Eigenvalues and Eigenvectors (3 weeks)**

The purpose of this chapter is to extend the work of the previous chapter to better understand the behavior of linear transformations, in particular with respect to identifying the image of $(T)$.

- Compute Eigenpairs for $2 \times 2$ and $3 \times 3$ matrices.
- Find bases for $\text{Dom}(T)$ using eigenvectors
- Geometrically describe $\text{Im}(T)$ using eigenpairs
- Find bases for $\text{Im}(T)$ using eigenvectors
- Implications of zero eigenvalue to $\text{Im}(T)$
- Diagonalize $2\times2$ and $3\times3$ matrices using eigenpairs
  - Diagonalize $A$, and compute $A^n$ (optional)
  - Symmetric Matrices
  - Quadric Forms (Optional)
Suggested assessment problems for Math 311.

Systems of Equations – Choose one of the following.

1. If \[
\begin{bmatrix}
1 \\
0 \\
3
\end{bmatrix}
\] is a solution to \(AX=B\) and \[
\begin{bmatrix}
0 \\
0 \\
t
\end{bmatrix}
\] is a solution to \(AX = 0\) for each real number \(t\), find all solutions to \(AX = B\).

2. Suppose \(A\) is row equivalent to \[
\begin{bmatrix}
0 & 0 & 0 & 2 & 1 & 0 \\
0 & 1 & 2 & 0 & 0 & 0 \\
1 & 2 & 1 & 0 & 0 & 0
\end{bmatrix}
\] and \[
\begin{bmatrix}
1
\end{bmatrix}
\] is a solution to \(AX = B\). Find all the solutions to \(AX=B\).

Rubric – one point for each of the following:
- Solutions to \(AX = 0\)
- Adding the one solution to the solutions to \(AX = 0\), even if those are incorrect.
- Correct solutions to \(AX = B\)

Vector Spaces -- Choose one of the following. In each, determine if the subset, with the inherited operations, forms a vector space.

1. \(V = \{(x, y, z) \in \mathbb{R}^3 : z = x + 2y\}\), under the usual operations in \(\mathbb{R}^3\).

2. \(V = \{f \in P_3 : f(0) = 0, f(1) = a\} \subset \mathbb{P}^3\), under the usual operations in \(P_3\), the vector space of polynomials over \(\mathbb{R}\) of degree less than or equal to 3.

Rubric – one point for each of the following.
- Suppose \(f, g\) are in \(V\) and \(c\) is in \(\mathbb{R}\).
- Check if \(f+g\) at 1 = 0 and \(f+g\) at 2 = 0
- Conclude that \(V\) is closed under addition.
- Check if \(V\) is closed under scalar multiplication.
- Don’t make any logic mistakes

3. \(V = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in M_2 : a, b, c, d \geq 0 \right\}\), under the usual operations in \(M_2\).

Linear Independence -- Choose one of the following.

1. Find a set of three linearly dependent vectors in \(\mathbb{R}^3\) that contain the vectors \((1, 0, 1)\) and \((1, 1, 1)\).
Rubric – One point for each

- Find a set
- Show it works

2. Determine whether or not the three vectors \((1, 0, 0), (0, 1, 0), (0, 1, 1)\) are linearly independent. Show your work and/or explain your answer.

Rubric – One point for each

- Set up matrix
- Find determinant or row reduce
- Correct explanation

Bases and Span. Choose one of the following.

1. Find a basis for the subspace of \(M_3\) defined as 
   \[
   \begin{bmatrix}
   x & 0 & 0 \\
   0 & x & 0 \\
   0 & 0 & x
   \end{bmatrix} : x \in \mathbb{R}
   \] 
   (Or you can replace the diagonal with \(x, 2x,\) and \(3x\)).

   Rubric – One point for each

   - Are the basis points in the space?
   - Do they involve diagonal matrices?
   - Correct?

2. Find a basis for \(V = \text{span of } \{f_1, f_2, f_3\}\), where each \(f_i\) is in \(P_2\), the space of polynomials of degree less than or equal to 2 with real coefficients, where \(f_1(x) = x, f_2(x) = x^2,\) and \(f_3(x) = 2x^2 + 3x\). Also, is \(V = P_2\)?

3. Let \(f, g \in P\), the vector space of polynomials with real coefficients, such that \(f\) and \(g\) are defined by \(f(x) = x^3\) and \(g(x) = x + 1\). Circle all those functions below that are elements of \(\text{span}\{f, g\}\).

   \[
   \begin{align*}
   5x^3 - x^2 + x + 1 & \quad x^3 - \pi x + \sqrt{2} \\
   (x + 1)^2 & \quad 3x + 3 \\
   6x^3 + 1 &
   \end{align*}
   \]

Linear transformations – Choose one of the following to assess determining whether or not an operator is linear.

1. If \(T : \mathbb{R}^2 \to \mathbb{R}^2\) is defined by \(T(x, y) = (-y, x)\), is \(T\) linear? Show work or explain.

2. If \(T : P_2 \to P_1\), where \(P_i\) is the vector space of polynomials of degree less than or equal to \(i\) with real coefficients, and \(T\) is defined by \(T(ax^2 + bx + c) = 2ax + b\), is \(T\) linear? Show work or explain.

Assess applying linearity. Choose one of the following.
1. If \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) is a linear transformation such that \( T(1, 0) = (3, 4) \) and \( T(0, 1) = (-1, 1) \), then compute \( T(1, 2) \).

2. If \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) is linear does \( T \) necessarily send every triangle to a triangle? Explain.

**Eigenvectors/Eigenvalues – computational**

1. Compute the eigenpairs for your favorite \( 2 \times 2 \) matrix (favorite for the instructor of course)

   **Rubric**
   - Set up determinant
   - Compute eigenvalues
   - Set up computing eigenvectors, even if for incorrect eigenvalues
   - Correct eigenpairs

**Eigenvectors/Eigenvalues – conceptual**

1. Suppose \( A \) is a \( 2 \times 2 \) matrix, \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) is defined by \( T(X) = AX \), and
   
   \[
   \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}
   \]

   are eigenpairs of \( A \). Geometrically describe the image of \( T \).
Math 431-432 Course Outline
Real Analysis

Overview: Real Analysis is a one-year course that is intended to expose advanced Math majors to the underlying theory of The Calculus, and to greatly increase their logic-based problem solving capabilities. It is offered in alternating years and is generally taken by junior or senior math majors.

Prerequisites: Math 206, 310, 311 or consent of the instructor.

Use of Technology

Math 431-432 is essentially a theorem-proving course and hence the use of technology is usually very limited, used perhaps to investigate the behavior of certain functions (e.g. recursive functions, fractals, etc.)

Primary Topics

Topology of the line (open/closed sets, compactness, l.u.b. axiom, limit points, dense/nowhere dense, separable, Cauchy)
Sequences/subsequences (monotonic, limits)
Continuity (at a point, on countable dense subsets, uniform continuity, uniform convergence, image of a compact set is compact, intermediate value theorem, etc.)
Cardinality (countable/uncountable, Baire’s Category Theorem)
Differentiability (mean value theorem,
Integration (Riemann, Riemann-Stieltjes, Bounded Variation)
Topology of $\mathbb{R}^n$
Metric Spaces (Complete metric spaces, topology)

Textbooks

This course can be taught with or without the use of a textbook. A typical set of notes might be based, for example, on *The Principles of Mathematics* by Rudin.

Student Learning Outcomes

Students completing this course should be able to:

- Demonstrate an understanding of the basic theorems and their implications regarding the topology of the line (and Euclidean n-space if they take 432), sequences and subsequences, compactness, denseness, convergence, continuity, differentiability, the definite integral, Cardinality, and metric spaces.
- Utilize a variety of standard theorem proving techniques to construct valid proofs, presented in a logically correct order (e.g. defining variables before using them).
- Construct counter-examples to false statements, and when feasible conjecture hypothesis that would make the statements true.
• Read and decipher previously unknown (to the student) definitions and readily apply them in straightforward cases.
• Determine if a prospective proof is valid. This includes identifying false or improper statements.

Sample set of questions/notes (Note: the following notes purposefully include incorrect statements, to which the students are supposed to provide corrections.)

Math 431

We begin Math 431 by considering problems concerning the set, \( \mathbb{R} \), of real numbers. These numbers we consider to be in one to one correspondence with points on a line, ordered from left to right in the usual way. We will assume all the familiar arithmetic and order. Subsets of the numbers may be defined by statements involving arithmetic and order. For example:

\[ S = \{ x : x \text{ is in } \mathbb{R} \text{ and } 2 + x > 4 \} \]

is the set to which \( x \) belongs if and only if \( x \) is to the right of 2 on the number line.

**Def.** The number set, \( A \), is a right ray means that if \( x \) is in \( A \) and \( y > x \), then \( y \) is in \( A \). We similarly define a left ray.

In addition, we will assume that the following statement holds for the numbers. We call this statement an Axiom and emphasize that it is an **assumption** which does not follow from the usual properties of arithmetic and order. Nevertheless, it is a reasonable property for the points on a line to have and we wish to assume that the set of real numbers also has it.

**Axiom 1** If \( \mathbb{R} \) is the union of the nonempty left ray \( A \) and the nonempty right ray \( B \), and \( A \) and \( B \) do not intersect, then either \( A \) has a largest element or \( B \) has a smallest element.

If each of \( a \) and \( b \) is a number such that \( a < b \), then we will use the following notation and terminology.

**open interval** \( (a,b) = \{ x : x \text{ is a number and } a < x \text{ and } x < b \} \)

**half-open interval** \( [a,b) = \{ x : x \text{ is a number and } a \leq x \text{ and } x < b \} \)

**half-open interval** \( (a,b] = \{ x : x \text{ is a number and } a < x \text{ and } x \leq b \} \)

**closed interval** \( [a,b] = \{ x : x \text{ is a number and } a \leq x \text{ and } x \leq b \} \)

**Def.** The statement that the number, \( p \), is a limit point of the number set, \( A \), means that if \( (a,b) \) is an open interval containing \( p \), (that is: \( a < p < b \)) then there is a number \( q \) such that \( a < q < b \), \( q \) is in \( A \), and \( q \) does not equal \( p \).

A more concise statement of the previous definition may read: \( p \) is a limit point of \( A \) if and only if each open interval containing \( p \) contains a number in \( A \) different from \( p \).
**Def.** The statement that the number set, A, is **closed** means that if p is a limit point of A then p is in A.

**Def.** The statement that the number set, A, is **open** means that if p is in A then there is an open interval containing p which is contained in A.

**Def.** The statement that f is a **function** means that f is a collection, each member of which is an ordered pair, no two of which have the same first coordinate. The set of first coordinates for f is called the **domain** of f, while the set of second coordinates is called the **image** of f.

**Def.** The statement that S is a **sequence** means that S is a function with domain some initial segment of the positive integers. (That is: the domain of S is either the set of positive integers or the domain of f is the set \( \{1,2,3,\ldots,n\} \) for some positive integer n.)

**Def.** The statement that p is the limit of the sequence S means that if (a,b) is an interval containing p, then there is a positive integer N such that S(i) is in (a,b) for each positive integer i > N.

**Def.** The statement that T is a **subsequence** of the sequence S means there is an increasing sequence, I, of positive integers such that T = S(I).

**Problems:**

1. There is a number set, A, such that 0 is a limit point of A.
2. Find a positive (without using the words none or no) statement that means that the set A is infinite.
3. If A is a number set and A has a limit point, p, then A is infinite.
4. If p is a limit point of A, then p is in A.
5. If A is infinite, then A has a limit point. Shown to be false.
5a. If A is uncountable, then A has a limit point.
5b. If A is infinite and unbounded, then A has a limit point.
6. The closed interval \([0,1]\) is infinite.
7. If c>0 there is a positive integer N such that \((1/N)<c\).
8. There is a number set, A, which has the property that A contains no open interval and each point in A is a limit point of A.
9. There is a closed number set, B, which satisfies the properties of A in problem 8.
10. There is a number, p, such that \(p^2=2\).
11. There is a number set that is neither open nor closed.
12. There is a number set that is both open and closed.
12b. There is a number set, other than \(\mathbb{R}\), which is both open and closed.
14. The rational numbers in \([0,1]\) form a countable set.
15. There is a set with exactly one limit point.
16. If p is a limit of the sequence S, then p is a limit point of Im(S).
17. A is open iff \(\mathbb{R} - A\) is closed.
18. If each of p and q is a limit of the number sequence S, then p = q.
19. If A is a collection of open number sets, then \(\bigcup_{x \in A} X\) is open.
20. If A is a collection of closed number sets, p is a number satisfying: if x is in A then p is in x, then \(\bigcap_{x \in A} x\) is closed.
The statement that the sequence \{A_i\} of number sets is **monotonically non-increasing** means if \(i\) is a positive integer then \(A_i \subseteq A_{i+1}\). We similarly define **monotonically non-decreasing**.

21. If \{A_i\} is a monotonically non-increasing sequence of intervals, then there is a \(p\) such that if \(i\) is a positive integer then \(p\) is in \(A_i\).

**DEF** The statement that the set \(A\) is **countable** means there is a surjective \(f : \mathbb{Z}^+ \rightarrow A\).

22. If each of \(A\) and \(B\) is countable, then \(A \cup B\) is countable.

**DEF** l.p.2 The statement that \(p\) is a **limit point of the number set** \(A\) means if \(c > 0\) there is a \(q\) in \(A\) such that \(q\) is not equal to \(p\) and \(|p - q| < c\).

23. \(p\) is a l.p.1 of the number set \(A\) iff \(p\) is l.p.2 of \(A\).

24. If \(p\) is a number, then there is a sequence \(S\) that has limit \(p\).

25. Find the limit point(s) of the following sets.

\[ A = \{1 + 1/2 + 1/3 + \ldots + 1/n : n\ is\ a\ positive\ integer\} \]

\[ B = \{e(1)^*1 + e(2)^*1/2 + \ldots + e(n)^*1/n : n\ is\ a\ positive\ integer\ and\ e(i)\ is\ in\ \{1,-1\}\ for\ each\ i\} \]

\[ C = \{n^*(2.5)^{\text{mod} 1} : n\ is\ a\ positive\ integer\} \]

Let \(a\) be a number, \(f\) be the real valued function defined by \(f(x) = x^2 - 2\), and define \(S_a = \{f^n(a) : n\ is\ a\ positive\ integer\}\). Here \(f^1(x) = f(x)\) and \(f^n(x) = f(f^{n-1}(x))\).

26. Axiom 1 implies the least upper bound axiom.

27. There exists a sequence \(S\) such that \(S\) has a limit and \(\text{Im}(S)\) has exactly 2 limit points.

28. If \(S\) is a monotonically increasing number sequence which is bounded above, then \(S\) has a limit.

29. If \(S\) is a sequence which satisfies if \(c > 0\) there is a positive integer \(N\) such that if \(i, j > N\) then \(|S(i) - S(j)| < c\), then \(S\) has a limit.

30. If \(A\) is a subset of the numbers and \(A\) is not closed then the set \(B = \{x : x\ is\ in\ A,\ or\ x\ is\ a\ limit\ point\ of\ A\}\) is closed. \(B\) is called the closure of \(A\).

31. If \(T\) is a sequence with limit \(0\), \(p\) is a number, and \(S\) is a sequence defined by \(S(n) = T(n) + p\), then \(S\) has limit \(p\).

32. If \(A\) is a number set and \(p\) is a limit point of \(A\), then there is a sequence \(S\) such that \(S(n)\) is in \(A\) for each \(n\) and \(S\) has limit \(p\).

33. Suppose for each positive integer \(i\), \((a(i),b(i))\) is an open interval such that if \(x\) is in \([0,1]\) then \(x\) is in \((a(i),b(i))\) for some \(i\). In this case we say that this collection, \(G\), of open intervals **covers** \([0,1]\). Then some **finite** subcollection of \(G\) covers \([0,1]\).

**Def.** The statement that \(T\) is a subsequence of the sequence \(S\) means there is an increasing sequence \(I\) of integers such that \(T = S \circ I\).

34. The sequence \(S\) has limit \(p\) iff each subsequence \(T\) has limit \(p\).
35. If the sequence $S$ is bounded, then $S$ has a subsequence $T$ which has a limit.

**Def.** If $D$ is a subset of $\mathbb{R}$ and $f$ is a real valued function with domain $D$ then the statement that $f$ is continuous at the point $p$ in $D$ means if $(a,b)$ contains $f(p)$, then there is a $(c,d)$ containing $p$ such that $f(x)$ is in $(a,b)$ for each $x$ in $D$ intersect $(c,d)$.

**Def.** If $D$ is a subset of $\mathbb{R}$, then the statement that $U$ is open (closed) relative to $D$ means there is an open(closed) set $V$ such that $U$ is the intersection of $V$ with $D$.

36. Suppose $D$ is a subset of $\mathbb{R}$, $f$ is a real valued function with domain $D$, and $p$ is in $D$. The following statements are equivalent:

a. $f$ is continuous.

b. if $p$ is in $D$ and $c>0$ then there is a $d>0$ such that if $x$ is in $D$ and $|x-p|<d$ then $|f(x)-f(p)|<c$.

c. if $U$ is an open set, then $f^{-1}(U)$ is open relative to $D$.

d. if $K$ is a closed set, then $f^{-1}(K)$ is closed relative to $D$.

e. if $p$ is in $D$ and $S$ is a sequence in $D$ with limit $p$, then the sequence $T$ defined by $T(n)=f(S(n))$ has limit $f(p)$.

37. Is the set $J$ countable?

38. What is the measure of the set $J$?

39. The sum of continuous functions is continuous over a common domain.

40. The product of continuous functions is continuous over a common domain.

41. The composition of continuous functions is continuous provided the domain for the outside function contains the image of the inside function.

42. If $G$ is a sequence of open sets, each dense in $\mathbb{R}$, then the intersection of $G$ is dense in $\mathbb{R}$.

43. If $F$ is a sequence of closed and nowhere dense sets, then the union of $F$ contains no interval.

44. $[0,1]$ is uncountable.

45. There is a function defined on $\mathbb{R}$ that is continuous at a single point only.

46. If $f$ is continuous over $U$ and $V$ is contained in $U$, then $f$ restricted to $V$ is continuous.

47. If $f$ is continuous over a compact set, then $\text{Im}(f)$ is compact.

48. If $f$ is continuous over $[a,b]$ and $f(a)<y<f(b)$, then there is a $c$ in $(a,b)$ such that $y=f(c)$.

49. If $f$ is continuous over the compact set $A$ and $c>0$, then there is a $d>0$ such that if each of $x$ and $y$ is in $A$ and $|x-y|<d$, then $|f(x)-f(y)|<c$.

50. There is a function $f$, defined over $\mathbb{R}$, such that $f$ is continuous at each irrational but discontinuous at each rational.

51. If $S$ is a sequence into the compact set $A$ then the sequence $T$ whose $n$th term is the average of the first $n$ terms of $S$ has a limit in $A$.

52. If for each positive integer $i$, $f_i$ is a continuous function defined over $[0,1]$ and for each $x$ in $[0,1]$ the sequence $f_i(x)$ converges to say, $f(x)$, and if then $f$ is continuous over $[0,1]$.

53. Is there a fat Cantor set?
54. If $c > 0$ there is an open cover of the rational numbers with length less than $c$.
55. $(0,1)$ is homeomorphic to $\mathbb{R}$. $[0,1]$ is not.
56. What is the minimum criteria on the construction for two sets to be homeomorphic, provided each is built by "taking out" open intervals?
57. There is a continuous function defined on $[0,1]$ which is neither increasing nor decreasing over a subinterval of $[0,1]$.

**DEF** The statement that the set $A$ is dense in $B$ means that each point in $B$ is either in $A$ or is a limit point of $A$.

**DEF** The statement that the set $A$ is nowhere dense in $B$ means that if $X$ is an open set relative to $B$ then $A$ is not dense in $X$.

**DEF** Suppose $f: [a,b] \to \mathbb{R}$ and $x \in (a,b)$. The statement that $f$ has slope at $x$ means there is a number $m$ such that the following function is continuous at $x$.

$$g(y) = \begin{cases} \frac{f(x) - f(y)}{x - y}, & y \in [a,b] \\ m, & y = x \end{cases}$$

85. If each of $f$ and $g$ has slope at $p$ then:
   a. $f+g$ has slope at $p$.
   b. $fg$ has slope at $p$.
86. If $g$ has slope at $p$ and $f$ has slope at $g(p)$ then $f \circ g$ has slope at $p$.
87. Is there a function defined on $[0,1]$ with slope at exactly one point?
90. Is there a $f: [0,1] \to \mathbb{R}$ such that $f$ has slope at each $x$ in $[0,1]$ but is not continuous on $[0,1]$?
91. If $f: [0,1] \to \mathbb{R}$ and $f$ has a max at $c$ then either $f$ has no slope at $c$ or the slope of $f$ at $c$ is 0.
92. If $f$ has slope on all of $[0,1]$ and $f(0)=f(1)$, then there is an $x$ in $(0,1)$ such that the slope of $f$ at $x$ is 0.
93. If $f$ has slope on $[a,b]$ then there is a $c$ in $(a,b)$ such that

$$f(c)(b-a) = f(b) - f(a), \text{equivalently, } f'(c) = \frac{f(b) - f(a)}{b-a}.$$ 

### 432 PROBLEMS

(Note: ignore numbering and repetition)

61. If $D$ is closed and bounded and $f: D \to \mathbb{R}$ and $\epsilon > 0$, then there exists a $\delta > 0$ such that if $x, y \in D$ and $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.
62. If $f$ is a nondecreasing, real valued function defined on $[0,1]$ and $f$ is not continuous, then $\{x \in [0,1]: f$ is discontinuous at $x\}$ is countable.
63. If $f:D \to \mathbb{R}$, $D$ is compact, $f$ is continuous, and $c>0$, then there is a $d>0$ such that if each of $x$ and $y$ is in $D$ and $|x-y|<d$ then $|f(x)-f(y)|<c$. 
64. There is a sequence of continuous functions that converge pointwise to a non-continuous function.

65. Suppose for each positive integer i, \( f_i \) is a continuous function defined over a common domain \( D \) and for each \( x \) in \( D \) the number sequence \( f_i(x) \) converges. If \( f: D \to \mathbb{R} \) defined by \( f(x) = \lim_{i \to \infty} f_i(x) \), and if the sequence \( f_i \) converges uniformly to \( f \), then \( f \) is continuous.

**Def** Let \( D \) be a subset of the reals and for each positive integer \( i \), let \( f_i: D \to \mathbb{R} \). The statement that this sequence of functions converges pointwise means that the number sequence \( S_p \) defined by \( S_p(n) = f_i(p) \) has limit for each \( p \) in \( D \).

**Def** If \( n \in \mathbb{Z}^+ \), \( p \in \mathbb{R}^n \) (Euclidean space of \( n \)-tuples, with the usual definition of distance), and \( \epsilon > 0 \), the \( \epsilon \)-ball centered at \( p \) is defined as the set of points in \( \mathbb{R}^n \) within \( \epsilon \) of \( p \).

66. Prove the following theorems for Euclidean \( n \)-space. (6, 29, 53, 61)

67. Is there a continuous \( f:[0,1] \to \mathbb{R} \) such that if \( (a,b) \) is contained in \([0,1]\) then there are \( w,x,y,z \) in \((a,b)\) such that \( w < x \), \( y < z \), \( f(w) < f(x) \), and \( f(y) > f(z) \) ?

**Def** If each of \( A \) and \( B \) are number sets, then the statement that \( A \) is homeomorphic to \( B \) means there is a continuous bijective \( f: A \to B \).

68. What is the minimum criterium on the construction for two Cantor sets to be homeomorphic, provided each is built by "taking out" open intervals?

**Def** Suppose \( f \) is a real valued function defined on some subset \( D \) of the reals. The statement that \( f \) has slope at \( p \) means that \( p \) is in \( D \), \( p \) is a limit point of \( D \), and there is a number \( m \) such that the function \( H \) defined on \( D \) by

\[
H(x) = \begin{cases} 
    \frac{f(x) - f(p)}{x - p}, & x \neq p \\
    m, & x = p 
\end{cases}
\]

is continuous at \( p \).

**Def** If \([a,b]\) is a closed interval, then by a subdivision of \([a,b]\) we will mean a finite collection \( D \) of non-overlapping closed intervals whose union is \([a,b]\). (Two closed intervals are nonoverlapping provided either they do not intersect or their intersection is a single point.)

**Def** The statement that the subdivision \( D \) refines the subdivision \( D' \) means that if \( d \) is in \( D \) then there is an \( e \) in \( D' \) such that \( d \) is a subset of \( e \). In this case we say that \( D \) is a refinement of \( D' \).
**Def** If \( D \) is a collection of sets, then a *choice function* for \( D \) is a function \( ch \) from \( D \) into the union of \( D \) with the property that \( ch(d) \) is in \( d \) for each \( d \) in \( D \).

**Def** If \( D \) is a subdivision of \([a,b]\), then the *mesh* of \( D \) is \( \max \{ |d| : d \text{ is in } D \} \). (\(|d|\) denotes the length of \( d \).)

**Def** Suppose \([r,s]\) is a closed interval and \( g \) is a real valued function whose domain contains \([r,s]\). the *\( g \)-length* of \([r,s]\) is the number \( g(r) - g(s) \) and will be denoted by \( g[r,s] \).

We are now in a position to define two (possibly different) notions of integral. Suppose \( I = [a,b] \) and each of \( f \) and \( g \) is a real valued function defined on \( I \).

**Def** The statement that \( f \) is *integrable on \( I \) with respect to \( g \)* means there is a number \( m \) such that if \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that if \( D \) is a subdivision of \( I \) with mesh less than \( \delta \) and \( ch \) is a choice function for \( D \), then

\[
\sum_{d \in D} f(ch(d)) \cdot g |d - m| < \varepsilon.
\]

In this case we will denote the number \( m \) by \( \int_a^b f \ dg \).

**Def** The statement that \( f \) is *type-R integrable on \( I \) with respect to \( g \)* means there is a number \( m \) such that if \( \varepsilon > 0 \) there is a subdivision \( D' \) of \( I \) such that if \( D \) refines \( D' \) and \( ch \) is a choice function for \( D \) then

\[
\sum_{d \in D} f(ch(d)) \cdot g |d - m| < \varepsilon.
\]

In this case we will denote the number \( m \) by \( \int_a^b f \ dg \).

69. If \( f \) is integrable on \( I \) with respect to \( g \) then the number \( m \) is unique.

70. Are the two types of integrals the same? In particular, if \( f \) is both integrable and type-R integrable, must the integrals be equal? Are there \( f, g, \) and \( I \) such that \( f \) is integrable with respect to \( g \) on \( I \) but not type-R integrable, or vice-versa?

71a. \( \int_a^b cf \ dg = c \int_a^b f \ dg \).

b. \( \int_a^b (f + g) \ dh = \int_a^b f \ dh + \int_a^b g \ dh \).

c. \( \int_a^b f d(g + h) = \int_a^b f \ dg + \int_a^b f \ dh \).
d. \[ \int_a^b f \, dg = c \int_a^b f \, dg. \]

72. If \( f \) has slope at \( p \), then \( f \) is continuous at \( p \).

72. Find a function \( f:[0,1] \to \mathbb{R} \) such that \( f \) is not integrable.

73. If \( \int_a^b f \, dg \) exists then \( f \) is continuous.

**Def** The statement that the function \( f:[a,b] \to \mathbb{R} \) is of **bounded variation** (denoted by B.V.) means that there is a number \( M > 0 \) such that if \( x_1, x_2, \ldots, x_n \) is a subdivision of \([a,b]\) then

\[
\sum_{i=1}^{n-1} |f(x_{i+1}) - f(x_i)| < M.
\]

74. There is a continuous function defined on \([0,1]\) which is not B.V.

75. If \( f:[0,1] \to \mathbb{R} \) is non-decreasing, then \( f \) is B.V.

76. If \( f \) is B.V. on \([0,1]\) then there are non-decreasing functions \( h \) and \( g \), each defined on \([0,1]\), such that \( f = h - g \).

77. If \( f:[0,1] \to \mathbb{R} \) is B.V. and \( x \) is in \([0,1]\), then \( f|_{[0,x]} \) is B.V.

78a. \[ \int_0^1 f \, dx \] exists for each continuous \( f \).

b. \[ \int_0^1 f \, dg \] exists for each continuous \( f \) and non-decreasing \( g \).

c. If \( f:[0,1] \to \mathbb{R} \) is continuous and \( g:[0,1] \to \mathbb{R} \) is B.V., then \( \int_0^1 f \, dg \) exists.

64. a) Find a function \( f: [0,1] \to \mathbb{R} \) such that \( f \) is not integrable.

b) If \( f \) is \([0,1] \to \mathbb{R} \) and discontinuous at each \( x \in [0,1] \) then \( \int_0^1 f \, dg \) does not exist.

65. If \( \int_a^b f \, dg \) exists then \( f \) is continuous.

67. Show that the salt and pepper function is not Riemann integrable.

70. If \( f:[0,1] \to \mathbb{R} \) and \( a \in (0,1) \) and each of \( f_{[0,a]} \) and \( f_{[a,1]} \) is B.V. then \( f \) is B.V.

72. Find a sequence of functions defined on \([0,1]\) that converge pointwise to a discontinuous function.

73. If the sequence of functions \( \{f_i\} \) converges pointwise to \( f \) and \( f \) is continuous, then \( f_i \) is continuous for each \( i \).
74. If the sequence of functions \( \{ f_i \} \) converges pointwise to \( f \) and \( f \) is continuous, then \( \{ f_i \} \) converges uniformly to \( f \).

75. Is there a function \( f: [0,1] \rightarrow [0,1] \) (non-decreasing) such that it is continuous and has slope 0 on the complement of the Cantor set?

76. If \( f:[0,1] \rightarrow \text{non-negative reals} \) is continuous and is Riemann integrable, \( f(\frac{1}{2}) = 0 \), \( \int_0^1 f \, dg \) exists (\( g \) is non-decreasing), then \( \int_0^1 f \, dx > 0 \).

77. Show \( R_2 \) and taxicab metrics are equivalent.

78. Find one that isn't.

79. If \( f:[0,1] \rightarrow \mathbb{R} \) is B.V., then there are non-decreasing \( g:[0,1] \rightarrow \mathbb{R} \) and \( h:[0,1] \rightarrow \mathbb{R} \) such that \( f = g - h \).

80. Suppose \( D \) is a dense subset of \([0,1]\) and \( f:D \rightarrow \mathbb{R} \) is uniformly continuous. Then there exists a unique \( g:[0,1] \rightarrow \mathbb{R} \) (continuous) such that \( g_D = f \).

81. Show that the function of #75 is the limit of a sequence of functions which converge uniformly.

82. Is a tangent-type function, unbounded, with symmetric positive and negative values, integrable?

83. Is \( \frac{1}{4} \) in the standard Cantor set?

**Def.** The statement that the set \( V \) is linearly independent means if \( \{ v_1, v_2, \ldots, v_n \} \subseteq V \) and \( c_1 v_1 + c_2 v_2 + \ldots + c_n v_n = \vec{0} \), then \( c_1, c_2, \ldots, c_n = 0 \).

**Def.** \( C[0,1] \) is the set of continuous functions defined over \([0,1]\)

84. Is \( C[0,1] \) finite dimensional?

85. \( \int_a^b f dg + \int_c^d f dg = \int_a^d f dg \).

86. If \( f:[0,1] \rightarrow \mathbb{R} \) is continuous and the function \( H:[0,1] \rightarrow \mathbb{R} \) defined by \( H(x) = \int_0^x f \, dx \), then \( H \) has slope at each \( x \) in \([0,1]\) and \( H'(x) = f(x) \).

87. Suppose \( g:[0,1] \rightarrow \mathbb{R} \) such that if \( f:[0,1] \rightarrow \mathbb{R} \) is continuous then \( \int_0^f f dg \) exists.

Suppose further that there is a number \( m \) such that if \( f([0,1]) \subseteq [-1,1] \) then \( \left| \int_0^f f dg \right| < m \).

Then \( g \) is B.V.

88. If \( g:[0,1] \rightarrow \mathbb{R} \) such that the slope of \( g \) is continuous on \([0,1]\), then \( g \) is B.V.

89. If each of \( f, g \) and the slope of \( g \) is continuous on \([0,1]\) then \( \int_0^f f dg = \int_0^f f' g dx \).
97. If \( \int_a^b f d\gamma \) exists, then \( \int_a^b g d\lambda \) exists.

**Def.** The ordered pair \((X,d)\) is a metric space means \(X\) is a set, \(d: X \times X \rightarrow \mathbb{R}^+\) satisfying:

i. \(d(x,y) \geq 0\) for each \(x, y\) in \(X\),
ii. \(d(x,y) = 0\) iff \(x = y\),
iii. \(d(x,y) \leq d(x,z) + d(z,y)\) for each \(x, y, z\) in \(X\).

In this case we say that \(d\) is a metric on \(X\).

98. Define \(d:C[0,1] \times C[0,1] \rightarrow \mathbb{R}^+\) by \(d(f, g) = \max \{|f(x) - g(x)|: x \text{ is in } [0,1]\}\)

a. Show that \((C[0,1], d)\) is a metric space.

b. Show that the function defined on \(R^2 \times R^2\) by

\[
d(x, y) = \begin{cases} 
\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}, & \text{quantity } \leq 1 \\
1, & \text{otherwise}
\end{cases}
\]

is a metric space for \(R^2\).

c. Show \((R^2, d)\) is topologically equivalent to \((R^2, D)\), where \(D\) is the usual metric distance on \(R^2\).

**Def.** \((X, T)\) is a topological space means \(X\) is a set, \(T\) is a collection of subsets of \(X\) such that

i. \(X \in T\), \(\emptyset \in T\);
ii. \(T; \cup\emptyset\) imply \(T; \cup\emptyset\); and iii. If \(A\) is a subset of \(T\), then \(\bigcup A \in T\).

**Def.** The statement that the topological space \((X, T)\) is topologically equivalent to the topological space \((X, S)\) means if \(A\) is a subset of \(X\) and \(p\) is in \(X\) then \(p\) is a limit point of \(A\) under the topology \(T\) if and only if \(p\) is a limit point of \(A\) under the topology of \(S\).

**Def.** The statement that \(A\) is separable means \(A\) contains a countable dense subset.

99. \(C[0,1]\) is separable.

100. \(R^2\) is separable.

101. Is \(R/Q\) separable?

102. If \(f\) is refinement-integrable with respect to \(g(x) = x\), then \(f\) is integrable with respect to \(g(x) = x\).

103. If \(f_i:[0,1] \rightarrow R\) is differentiable for each positive integer \(i\) and \(\{f_i\} \rightarrow f\) (uniformly ?) and \(\{f_i\} \rightarrow g\) uniformly, then \(f = g\).

104. Show that a metric on the set \(X\) induces a topology on \(X\).

**Def** The statement that the metric space \((X, d)\) is complete means that if \(S; Z^+ \rightarrow R\) is a Cauchy sequence in \(X\) then \(S\) has a limit \(p\) in \(X\).

105. Find a metric, \(d\), on \(R\) such that \((R, d)\) is not complete.

106. Show \((R^2, d)\), where \(d\) is the metric of \#98, is not compact.

107. Show that if \(A\) is a subset of a metric space \((X, d)\), and \(A\) is not closed, then \(A\) is not compact.
DEF The statement that the metric space is totally bounded means if \( c > 0 \) and \( B \) is an infinite subset of \( A \) then there are \( a \) and \( b \) in \( B \) such that \( d(a,b) < c \).

108. A metric space is compact iff it is complete and totally bounded.
109. \( f: [0,1] \rightarrow \mathbb{R} \) is integrable iff \( f \) is bounded and \( \{ x : f \text{ is discontinuous at } x \} \) has measure 0.
110. Suppose \( f: [0,1] \rightarrow \mathbb{R} \) is bounded and \( g: [0,1] \rightarrow \mathbb{R} \) is non-decreasing and continuous at each point of discontinuity of \( f \). Then \( f \) is \( g \)-integrable on \([0,1]\).

111. Is there and \( f: [0,1] \rightarrow \mathbb{R} \) such that \( f \) is bounded, \( f^2 \) is Riemann integrable, but \( f \) is not Riemann integrable?
Math 431

We begin Math 431 by considering problems concerning the set, \( \mathbb{R} \), of real numbers. These numbers we consider to be in one-to-one correspondence with points on a line, ordered from left to right in the usual way. We will assume all the familiar arithmetic and order and thus, subsets of the numbers may be defined by statements involving arithmetic and order. For example:

\[ \text{S} = \{ x : x \text{ is in } \mathbb{R} \text{ and } 2 + x > 4 \} \]

is the set to which \( x \) belongs if and only if \( x \) is greater than 2.

**Def.** The number set, \( A \), is a **right ray** means that if \( x \) is in \( A \) and \( y > x \), then \( y \) is in \( A \). We similarly define a left ray.

In addition, we will assume that the following statement holds for the numbers. We call this statement an Axiom and emphasize that it is an **assumption** that does not follow from the usual properties of arithmetic and order. Nevertheless, it is a reasonable property for the points on a line to have and we wish to assume that the set of real numbers also has it.

**Axiom 1** If \( \mathbb{R} \) is the union of the nonempty left ray \( A \) and the nonempty right ray \( B \), and \( A \) and \( B \) do not intersect, then either \( A \) has a largest element or \( B \) has a smallest element.

If each of \( a \) and \( b \) is a number such that \( a < b \), then we will use the following notation and terminology.

<table>
<thead>
<tr>
<th>Type</th>
<th>Notation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>open interval</strong></td>
<td>((a, b)) = {x : x \text{ is a number and } a &lt; x \text{ and } x &lt; b}</td>
<td></td>
</tr>
<tr>
<td><strong>half-open interval</strong></td>
<td>([a, b)) = {x : x \text{ is a number and } a \leq x \text{ and } x &lt; b}</td>
<td></td>
</tr>
<tr>
<td><strong>half-open interval</strong></td>
<td>((a, b]) = {x : x \text{ is a number and } a &lt; x \text{ and } x \leq b}</td>
<td></td>
</tr>
<tr>
<td><strong>closed interval</strong></td>
<td>([a, b]) = {x : x \text{ is a number and } a \leq x \text{ and } x \leq b}</td>
<td></td>
</tr>
</tbody>
</table>

**Def.** The statement that the number, \( p \), is a **limit point** of the number set, \( A \), means that if \((a, b)\) is an open interval containing \( p \), (that is: \( a < p < b \)) then there is a number \( q \) such that \( a < q < b \), \( q \) is in \( A \), and \( q \) does not equal \( p \).

A more concise statement of the previous definition may read: \( p \) is a limit point of \( A \) if and only if each open interval containing \( p \) contains a number in \( A \) different from \( p \).

**Def.** The statement that the number set, \( A \), is **closed** means that if \( p \) is a limit point of \( A \) then \( p \) is in \( A \).

**Def.** The statement that the number set, \( A \), is **open** means that if \( p \) is in \( A \) then there is an open interval containing \( p \) that is contained in \( A \).

**Def.** The statement that \( f \) is a **function** means that \( f \) is a collection, each member of which is an ordered pair, no two of which have the same first coordinate. The set of first coordinates for \( f \) is called the domain of \( f \), while the set of second coordinates is called the image of \( f \).

**Def.** The statement that \( S \) is a **sequence** means that \( S \) is a function with domain some initial segment of the positive integers. (That is: the domain of \( S \) is either the set of positive integers or the domain of \( f \) is the set \( \{1, 2, 3, \ldots, n\} \) for some positive integer \( n \).)
**Def.** The statement that \( p \) is the limit of the sequence \( S \) means that if \((a,b)\) is an interval containing \( p \), then there is a positive integer \( N \) such that \( S(i) \) is in \((a,b)\) for each positive integer \( i \geq N \).

**Def.** The statement that \( T \) is a subsequence of the sequence \( S \) means there is an increasing sequence, \( I \), of positive integers such that \( T = S(I) \).

**Def.** The statement that the function \( f : A \to B \) is surjective (another name is “onto”) means that if \( y \) is in \( B \) then there is an \( x \) in \( A \) such that \( f(x) = y \).

**Def.** The statement that the set \( A \) is *countable* means there is a surjective \( f : \mathbb{Z}^+ \to A \).

Problems:

1. There is a number set, \( A \), such that 0 is a limit point of \( A \).
2. If \( A \) is a number set and \( A \) has a limit point, \( p \), then \( A \) is infinite.
3. If \( p \) is a limit point of \( A \), then \( p \) is in \( A \).
4. If \( A \) is infinite, then \( A \) has a limit point.
   4b. If \( A \) is an infinite subset of \([0,1]\) then \( A \) has a limit point.
   4c. If \( A \) is uncountable then \( A \) has a limit point.
5. Find a positive (without using the words none or no) statement that means that the set \( A \) is infinite. (i.e. not finite is not what we’re looking for)
6. The closed interval \([0,1]\) is infinite.
7. If \( c > 0 \) there is a positive integer \( N \) such that \((1/N) < c\).
8. There is a number set, \( A \), which has the property that \( A \) contains no open interval and each point in \( A \) is a limit point of \( A \).
9. There is a closed number set, \( B \), which satisfies the properties of \( A \) in problem 8.
10. There is a number, \( p \), such that \( p \cdot p = 2 \).
11. There is a number set that is neither open nor closed.
12. There is a number set that is both open and closed.
13. The rational numbers in \([0,1]\) form a countable set.
14. There is a set with exactly one limit point.
15. Find the limit point(s) of the following sets.
   \[ A = \{1 + 1/2 + 1/3 + \ldots + 1/n : n \text{ is a positive integer} \} \]
   \[ B = \{e(1)*1+e(2)*1/2+\ldots+e(n)*1/n : n \text{ is a positive integer and } e(i) \text{ is in } \{1,-1\} \text{ for each } i \} \]
   \[ C = \{n*(2^5) \mod 1 : n \text{ is a positive integer} \} \]
   Let \( a \) be a number, \( f \) be the real valued function defined by \( f(x) = x^2 - 2 \), and define
   \[ S_a = \{f^n(a) : n \text{ is a positive integer} \}. \]
   Here \( f^1(x) = f(x) \) and \( f^n(x) = f(f^{n-1}(x)) = f \circ f^{n-1}(x) \).

16. Assuming \( A \) is not \( \mathbb{R} \), \( A \) is open iff \( \mathbb{R} - A \) is closed.

**Def.** The statement that the set \( B \) is the closure of the set \( A \) means that \( B \) is the union of \( A \) and the set of limit points of \( A \). (In the case \( A \) has no limit point, then the closure of \( A \) is simply \( A \).)

17. The closure of \( A \) is closed.

18. The statement that \( p \) is not a limit point of the set \( A \) means…

19. If \( p \) is a number, then there is a sequence \( S \) that has limit \( p \).

20. There exists a sequence \( S \) such that \( S \) has a limit and \( \text{Im}(S) \) has exactly 2 limit points.

   20a. If the sequence \( S \) has a limit then the limit is unique.
   20b. Can the \( \text{Im}(S) \) be uncountable?
   20c. Direct proof for number 20.


22. l.p. iff l.p.2 (l.p. is the open interval definition, l.p.2 uses distance)

23. If each of \( A \) and \( B \) is countable, then \( A \) union \( B \) is countable.

   a) The countable union of countable sets is countable.

**Axiom 2** If \( A \) is a number set and \( b \) is a number such that if \( x \in A \) then \( x \leq b \), then there is a number \( q \) such that if \( x \) is in \( A \) then \( x \leq q \) and if \( r < q \) then there is a \( y \) in \( A \) such that \( r < y \leq q \).


25. If \( A \) is a collection of open number sets, then \( \bigcup_{x \in A} X \) is open.

26. If \( A \) is a collection of closed number sets and \( p \) is a number satisfying: if \( x \) is in \( A \) then \( p \) is in \( x \), then \( \bigcap_{x \in A} x \) is closed.

27. If \( p \) is a limit of the sequence \( S \), then \( p \) is a limit point of \( \text{Im}(S) \). (Shown to be false, but improved to \( \text{Im}(S) \) is infinite.)
**DEF** The statement that the sequence \( \{A_i\} \) of number sets is *monotonically non-increasing* means if \( i \) is a positive integer then \( A_i \subseteq A_{i+1} \). We similarly define *monotonically non-decreasing*.

28. If \( \{A_i\} \) is a monotonically non-increasing sequence of intervals, then there is a \( p \) such that if \( i \) is a positive integer then \( p \) is in \( A_i \). Shown to be false, but true if intervals are closed.

28b. Replace closed intervals with compact sets.

**Def.** The statement that \( T \) is a subsequence of the sequence \( S \) means there is an increasing sequence \( I \) of integers such that \( I \circ S = T \).

29. If \( p \) is a limit point of \( \text{Im}(S) \), then there exists a subsequence \( T \) with limit \( p \).

30. \( .725123123123… = ? \)

31. If \( A \) is open and \( p \) is in \( A \) and there is a \( q > p \) such that \( q \) is not in \( A \) then there is a \( b \) not in \( A \) such that \( (p, b) \) is a subset of \( A \).

32. If \( A \) is countable then \( A \) has measure 0.

33. If \( A \) is open then there exists a sequence of open intervals \( I \) such that \( A = \bigcup I \).

**Def.** The statement that the number set \( A \) has measure zero means that if \( c > 0 \) there is a countable (possibly finite) collection of open intervals \( (a_i, b_i) \) such that \( A \subseteq \bigcup (a_i, b_i) \) and \( \sum b_i - a_i < c \).

34. \( T_1 \). Is \( \frac{1}{4} \) in the \( T \)-set? showed false if we use \( \pi / 4 \).

\( T_2 \). Is \( T \) countable?

\( T_3 \). Show that each point in \( T \) is a limit point of \( T \).

\( T_4 \). Show that \( T \) contains no open interval.

\( T_5 \). Show that \( T \) has measure 0.

\( T_6 \). Show there is a “fat” \( T \)-set.

\( T_7 \). Every “random” \( T \)-set is homeomorphic to the \( T \)-set.

35. Suppose for each positive integer \( n \), \( G_n \) is an open set with the property that if \( x \) is a number then \( x \) is a limit point of \( G_n \). Let \( G = \bigcap G_n \). Show that each number is a limit point of \( G \). (i.e. If \( G \) is a sequence of open dense sets then the intersection of the members
36. Suppose for each integer n, $F_n$ is a closed set that contains no open interval, and $F = \bigcup F_n$. Show that F is not equal to the numbers.

37. $[0,1]$ is uncountable.

38. If $S$ is a monotonically non-decreasing number sequence which is bounded above, then $S$ has a limit.

39. If $S$ is a sequence which satisfies if $c > 0$ there is a positive integer $N$ such that if $i, j > N$ then $|S(i) - S(j)| < c$, then $S$ has a limit.

40. Suppose $G$ is a collection of open intervals such that if $x$ is in $[0,1]$ then $x$ is in $g$ for some $g \in G$. In this case we say that $G$ is an open cover of $[0,1]$. Show there is a finite sub-collection of $G$ that also covers $[0,1]$. A set that has the property that each open cover has a finite subcover is called compact.

   a) Replace open interval with open set.
   b) A number set is compact iff it is closed and bounded.

41. The sequence $S$ has limit $p$ iff each subsequence $T$ has limit $p$.

42. If the sequence $S$ is bounded, then $S$ has a subsequence $T$ which has a limit.

43. If $T$ is a sequence with limit $t$ and $R$ is a sequence with limit $r$, then the sequence $S$ defined by $S(n) = T(n) + R(n)$ has limit $t + r$.

**Def.** If $D$ is a subset of $\mathbb{R}$ and $f$ is a real valued function with domain $D$ then the statement that $f$ is continuous at the point $p$ in $D$ means if $(a, b)$ contains $f(p)$, then there is a $(c, d)$ containing $p$ such that $f(x)$ is in $(a, b)$ for each $x$ in $D \cap (c, d)$.

**Def.** The statement that $f$ is continuous means that $f$ is continuous at each point in its domain.

**Def.** If $D$ is a subset of $\mathbb{R}$, then the statement that $U$ is open/closed relative to $D$ means there is an open/closed set $V$ such that $U = V \cap D$.

44. Suppose $D$ is a subset of $\mathbb{R}$, $f$ is a real valued function with domain $D$, and $p$ is in $D$. The following statements are equivalent:

   a. $f$ is continuous.
   b. if $p$ is in $D$ and $c > 0$ then there is a $d > 0$ such that if $x$ is in $D$ and $|x-p| < d$ then $|f(x) - f(p)| < c$.
   c. if $U$ is an open set, then $f^{-1}(U)$ is open relative to $D$.
   d. if $K$ is a closed set, then $f^{-1}(K)$ is closed relative to $D$.
   e. if $p$ is in $D$ and $S$ is a sequence in $D$ with limit $p$, then the sequence $T$ defined by
T(n)=f(S(n)) has limit f(p).

45. The sum of continuous functions is continuous over a common domain.
46. The product of continuous functions is continuous over a common domain.
47. The composition of continuous functions is continuous provided the domain for the outside function contains the image of the inside function.
48. There is a function defined on R that is continuous at a single point only.
49. There is a non-decreasing function f defined on [0,1] such that f is continuous at infinitely many points and discontinuous at infinitely many points.
50. If f is continuous over U and V is contained in U, then f restricted to V is continuous.
51. If f is continuous over D then there is an x in D such that f(x) ≥ f(y) for each y in D.
   Showed to be false.
   51a. If f is continuous over D and D is bounded then same conclusion. Shown to be false as well.
   51b. If f is continuous over the compact set D, then the image of f is compact.
52. If f is continuous over D and A is a closed subset of D then f(A) is closed. Shown to be false
   52a. replace “closed” with “compact”.
53. If f is continuous over D and f(a) < y < f(b), then there is a c in D such that y = f(c).
   Showed to be false
   53a. Replace arbitrary D with a closed interval.
54. If f is continuous over the compact set A and c > 0, then there is a d > 0 such that if each of x and y is in A and |x - y| < d, the |f(x) - f(y)| < c.
55. There is a function f, defined over [0,1] such that f is continuous on the irrational numbers and discontinuous on the rationals.
56. If \( f : [0,1] \to \mathbb{R} \) is non-decreasing and not continuous, then \{x in [0,1]: f is not continuous at x\} is countable.

57. There is an \( f : [0,1] \to \mathbb{R} \) such that f is continuous and f “cannot be drawn”. **Test 2 # 5 iv.**

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**Math 432**

Open problems from 431:

T7. Every “random” T-set is homeomorphic to the T-set.

42. If the sequence \( S \) is bounded, then \( S \) has a subsequence \( T \) which has a limit.

51b. The continuous image of a compact set is compact.

53a. If \( f \) is continuous over \([a,b]\) and \( f(a) < y < f(b) \), then there is a \( c \) in \( D \) such that \( y = f(c) \).

54. If \( f \) is continuous over the compact set \( A \) and \( c > 0 \), then there is a \( d > 0 \) such that if each of \( x \) and \( y \) is in \( A \) and \( |x - y| < d \), the \( |f(x) - f(y)| < c \).

55. a. There is a non-decreasing function \( f \), defined over \([0,1]\) such that \( f \) is continuous on the irrational numbers and discontinuous on the rationals.

56. If \( f : [0,1] \to \mathbb{R} \) is non-decreasing and not continuous, then \{x in [0,1]: f is not continuous at x\} is countable.

**New Problems**

**Def.** The statement that the set \( A \) is *homeomorphic* to the set \( B \) means there is a continuous bijection from \( A \) onto \( B \) (and vice-versa).

58. \((0,1)\) is homeomorphic to \( \mathbb{R} \). \([0,1]\) is not.

59. There is a continuous function defined on \([0,1]\) which is neither increasing nor decreasing (nor constant) over each subinterval of \([0,1]\).

60. If \( A \) is compact and \( f : A \to B \) is a continuous bijection, then \( f^{-1} \) is continuous.

   a. Why can’t we just use closed instead of compact?

61. If \( A \) is a dense subset of the \([0,1]\) and \( f : A \to \mathbb{R} \) is continuous, then there is a continuous \( g : [0,1] \to \mathbb{R} \) such that \( g \) restricted to \( A \) is \( f \).

**Def.** Suppose \( f : [a,b] \to \mathbb{R} \) and \( x \in (a,b) \). The statement that \( f \) *has slope at x* means there is a number \( m \) such that the following function is continuous at \( x \).
\[ g(y) = \begin{cases} \frac{f(x) - f(y)}{x - y}, & y \in [a, b] \\ \frac{m}{y - x}, & y = x \end{cases} \]

62. If each of \( f \) and \( g \) has slope at \( p \) then:
   a) \( f + g \) has slope at \( p \).
   b) \( fg \) has slope at \( p \).

63. If \( g \) has slope at \( p \) and \( f \) has slope at \( g(p) \) then \( f \circ g \) has slope at \( p \).

64. Is there a function defined on \([0,1]\) with slope at exactly one point?

65. If \( f \) has slope at \( p \), then \( f \) is continuous at \( p \).

66. If \( f: [0,1] \rightarrow \mathbb{R} \) and \( f \) has a max at \( c \) then either \( f \) has no slope at \( c \) or the slope of \( f \) at \( c \) is 0.

67. If \( f \) has slope on all of \([0,1]\) and \( f(0) = f(1) \), then there is an \( x \) in \((0,1)\) such that the slope of \( f \) at \( x \) is 0.

68. If \( f \) has slope on \([a,b]\) then there is a \( c \) in \((a,b)\) such that
   \[ f'(c)(b - a) = f(b) - f(a), \text{ equivalently, } f'(c) = \frac{f(b) - f(a)}{b - a} \]

**Def** Let \( D \) be a subset of the reals and for each positive integer \( i \), let \( f_i: D \rightarrow \mathbb{R} \). The statement that this sequence of functions converges pointwise means that for each \( p \) in \( D \) the number sequence \( S_p \) defined by \( S_p(n) = f_i(p) \) has limit.

69. There is a sequence of continuous functions that converge pointwise to a non-continuous function.

**Def** The statement that \( S \) is a metric space means there is a function \( d: \mathbb{S} \times \mathbb{S} \rightarrow \mathbb{R} \) such that
   i. \( d(s,s) = 0 \) for each \( s \) in \( S \)
   ii. \( d(s,r) \geq 0 \) for each \( r \) and \( s \) in \( S \)
   iii. \( d(r,t) \leq d(r,s) + d(s,t) \) for each \( r, s, \) and \( t \) in \( S \)

Note: The usual definition of distance between points in \( \mathbb{R}^2 \) and \( \mathbb{R}^n \) form a metric for those spaces. That is, the topology of the line can easily be extended to Euclidean 2-space (i.e. \( \mathbb{R}^2 \) or any other metric space for that matter, using the following definition of a basic open set).

**Def** If \( p \) is a point and \( \varepsilon \) is greater than zero, then the \( \varepsilon \) ball centered at \( p \) is defined to be the set of points within \( \varepsilon \) of \( p \). This is also referred to as the \( \varepsilon \) neighborhood of \( p \).
This definition naturally extends the topology of the line to that of the plane, and beyond. Of course, since we also have limit points defined in terms of distance, as long as we have the concept of distance, which we already do in Euclidean n-space, it is possible to mimic many of the theorems from 431 to higher dimensional spaces.

Also, the “positive number c definition” of limit convergence can be extended to each metric space. Axiom 1 from 431 essentially states that the real numbers do not contain a “hole”. A more instructive way of viewing this statement might be to say that each Cauchy Sequence in R converges to a point in R (problem 39).

70. Prove the following problems from 431 for Euclidean 2-space.
   a) If A is infinite and bounded, then A has a limit point (Problem 4b)
   b) $\mathbb{R}^2$ is complete. (see def below. Problem 39)
   c) The closed ball of radius 1 is compact. (Problem 40)
   d) The intersection of a monotonically non-increasing sequence of compact sets has a point in common (Problem 28b.)

**Def** The statement that the metric space S is complete means that each Cauchy Sequence in S converges to a point in S.

The results of problem 69 lead us to believe that if we wish to build a complete space of continuous functions over [0,1], we need to build a suitable metric, one that goes beyond pointwise convergence for example. In order to do this, think about what it might mean for 2 functions in $C[0,1]$ to be within epsilon of each other.

71. Define a metric on $C[0,1]$ such that $C[0,1]$ represents a Complete metric space.

72. There is an $f:[0,1] \rightarrow \mathbb{R}$ such that f has slope at each x in [0,1] but such that the derivative of f is unbounded.

73. $C[0,1]$ is separable.

74. Is there an $f:[0,1] \rightarrow \mathbb{R}$ such that f has slope at each x in [0,1] but its derivative is not continuous on [0,1]?

**Def** If [a,b] is a closed interval, then by a subdivision of [a,b] we mean a finite collection D of non-overlapping closed intervals whose union is [a,b]. (Two closed intervals are nonoverlapping provided either they do not intersect or their intersection is a single point.) An equivalent definition is that a subdivision of [a,b] is a finite increasing sequence in [a,b] such that its initial point is a and its final point is b.

**Def** The statement that the subdivision D refines the subdivision D’ means that if d is in D then there is an e in D’ such that d is a subset of e. In this case we say that D is a refinement of D’. (Equivalently using the sequence definition, D refines D’ means that D’ is a subsequence of D.)

**Def** If D is a collection of sets, then a choice function for D is a function ch from D into the union
of $D$ with the property that $c_h(d)$ is in $d$ for each $d$ in $D$.

**Def** If $D$ is a subdivision of $[a,b]$, then the *mesh* of $D$ is $\max \{|d|: d \text{ is in } D\}$. (\(|d|\) denotes the length of $d$.)

**Def** Suppose $[r,s]$ is a closed interval and $g$ is a real valued function whose domain contains $[r,s]$. The *$g$-length* of $[r,s]$ is the number $g(r)-g(s)$ and will be denoted by $g[r,s]$.

We are now in a position to define two (possibly different) notions of integral. Suppose $I=[a,b]$ and each of $f$ and $g$ is a real valued function defined on $I$.

**Def** The statement that $f$ is integrable on $I$ with respect to $g$ means there is a number $m$ such that if $\varepsilon > 0$ there is a $\delta > 0$ such that if $D$ is a subdivision of $I$ with mesh less than $\delta$ and $c_h$ is a choice function for $D$, then

$$\left| \sum_{d \in D} f(c_h(d)) \cdot g |d - m| \right| < \varepsilon.$$  In this case we will denote the number $m$ by $\int_a^b f \, dg$.

**Def** The statement that $f$ is type-R integrable on $I$ with respect to $g$ means there is a number $m$ such that if $\varepsilon > 0$ there is a subdivision $D$ of $I$ such that if $D$ refines $D$ and $c_h$ is a choice function for $D$ then

$$\left| \sum_{d \in D} f(c_h(d)) \cdot g |d - m| \right| < \varepsilon.$$  In this case we will denote the number $m$ by $\int_a^b f \, dg$.

(a) If $f$ is integrable on $I$ with respect to $g$ then the number $m$ is unique. *Aja*

(b) Are the two types of integrals the same? In particular, if $f$ is both integrable and type-R integrable, must the integrals be equal? Are there $f$, $g$, and $I$ such that $f$ is integrable with respect to $g$ on $I$ but not type-R integrable, or vice-versa?

(c) $\int_a^b cf \, dg = c \int_a^b f \, dg$.

(d) $\int_a^b (f + g) \, dh = \int_a^b f \, dh + \int_a^b g \, dh$.

(e) $\int_a^b f(d + h) = \int_a^b f \, dg + \int_a^b f \, dh$.

(f) $\int_a^b f \, dcg = c \int_a^b f \, dg$.

**Def** Find a function $f:[0,1] \to \mathbb{R}$ such that $f$ is not integrable.
77. If \( f: [0,1] \rightarrow \) non-negative reals is continuous, \( g \) is non-decreasing, \( f(\frac{1}{2}) > 0 \), and \( \int_0^1 f \, dx \) exists, then \( \int_0^1 f \, dx > 0 \). Is this true if we replace \( dx \) with \( dg \) for each non-decreasing \( g \)?

78. If \( \int_a^b f \, dg \) exists then \( f \) is continuous.

79. Is there an \( f: [0,1] \rightarrow \mathbb{R} \) such that \( f \) is bounded, \( f^2 \) is Riemann integrable, but \( f \) is not Riemann integrable?

**Def** The statement that the function \( f: [a,b] \rightarrow \mathbb{R} \) is of **bounded variation** (denoted by B.V.) means that there is a number \( M \geq 0 \) such that if \( x_1, x_2, \ldots, x_n \) is a subdivision of \([a,b]\) then

\[
\sum_{i=1}^{n-1} |f(x_{i+1}) - f(x_i)| < M.
\]

80. There is a continuous function defined on \([0,1]\) which is not B.V.

81. If \( f: [0,1] \rightarrow \mathbb{R} \) is non-decreasing, then \( f \) is B.V.

82. If \( f: [0,1] \rightarrow \mathbb{R} \) and \( a \in (0,1) \) and each of \( f_{[0,a]} \) and \( f_{[a,1]} \) is B.V. then \( f \) is B.V.

83. If \( f \) is B.V. on \([0,1]\) then there are non-decreasing functions \( h \) and \( g \), each defined on \([0,1]\), such that \( f = h - g \).

84. If \( f: [0,1] \rightarrow \mathbb{R} \) is B.V. and \( x \) is in \([0,1]\), then \( f_{[0,x]} \) is B.V.

85. a. \( \int_0^1 f \, dx \) exists for each continuous \( f \).

b. \( \int_0^1 f \, dg \) exists for each continuous \( f \) and non-decreasing \( g \).

c. \( \int_0^1 f \, dg \) exists for each continuous \( f \) and BV \( g \).

86. Show \( f(x) = \begin{cases} 0 & 0 \leq x < .5 \\ 1 & .5 \leq x \leq 1 \end{cases} \) is Riemann integrable over \([0,1]\).

87. There is a \( f: [0,1] \rightarrow \mathbb{R} \) such that \( f \) has infinitely many points of discontinuity and is Riemann integrable.

a. There is such an \( f \) that is non-decreasing.

b. There is such an \( f \) that is discontinuous at each rational.
Math 431

We begin Math 431 by considering problems concerning the set, \( \mathbb{R} \), of real numbers. These numbers we consider to be in one-to-one correspondence with points on a line, ordered from left to right in the usual way. We will assume all the familiar arithmetic and order and thus, subsets of the numbers may be defined by statements involving arithmetic and order. For example:

\[
S = \{x : x \text{ is in } \mathbb{R} \text{ and } 2 + x > 4\}
\]

is the set to which \( x \) belongs if and only if \( x \) is greater than 2.

**Def.** The number set, \( A \), is a right ray means that if \( x \) is in \( A \) and \( y > x \), then \( y \) is in \( A \). We similarly define a left ray.

In addition, we will assume that the following statement holds for the numbers. We call this statement an Axiom and emphasize that it is an assumption that does not follow from the usual properties of arithmetic and order. Nevertheless, it is a reasonable property for the points on a line to have and we wish to assume that the set of real numbers also has it.

**Axiom 1** If \( \mathbb{R} \) is the union of the nonempty left ray \( A \) and the nonempty right ray \( B \), and \( A \) and \( B \) do not intersect, then either \( A \) has a largest element or \( B \) has a smallest element.

If each of \( a \) and \( b \) is a number such that \( a < b \), then we will use the following notation and terminology.

- **open interval** \((a,b) = \{x : x \text{ is a number and } a < x \text{ and } x < b\}\)
- **half-open interval** \([a,b) = \{x : x \text{ is a number and } a \leq x \text{ and } x < b\}\)
- **half-open interval** \((a,b] = \{x : x \text{ is a number and } a < x \text{ and } x \leq b\}\)
- **closed interval** \([a,b] = \{x : x \text{ is a number and } a \leq x \text{ and } x \leq b\}\)

**Def.** The statement that the number, \( p \), is a limit point of the number set, \( A \), means that if \((a,b)\) is an open interval containing \( p \), (that is: \( a < p < b \)) then there is a number \( q \) such that \( a < q < b \), \( q \) is in \( A \), and \( q \) does not equal \( p \).

A more concise statement of the previous definition may read: \( P \) is a limit point of \( A \) if and only if each open interval containing \( p \) contains a number in \( A \) different from \( p \).

**Def.** The statement that the number set, \( A \), is closed means that if \( p \) is a limit point of \( A \) then \( p \) is in \( A \).

**Def.** The statement that the number set, \( A \), is open means that if \( p \) is in \( A \) then there is an open interval containing \( p \) that is contained in \( A \).

**Def.** The statement that \( f \) is a function means that \( f \) is a collection, each member of which is an ordered pair, no two of which have the same first coordinate. The set of first coordinates for \( f \) is called the domain of \( f \), while the set of second coordinates is called the image of \( f \).

**Def.** The statement that \( S \) is a sequence means that \( S \) is a function with domain some initial segment of the positive integers. (That is: the domain of \( S \) is either the set of positive integers or the domain of \( f \) is the set \{1,2,3,....,n\} for some positive integer \( n \).)
Def. The statement that $p$ is the limit of the sequence $S$ means that if $(a,b)$ is an interval containing $p$, then there is a positive integer $N$ such that $S(i)$ is in $(a,b)$ for each positive integer $i \geq N$.

Def. The statement that $T$ is a subsequence of the sequence $S$ means there is an increasing sequence, $I$, of positive integers such that $T = S(I)$.

Def. The statement that the function $f : A \to B$ is surjective (another name is “onto”) means that if $y$ is in $B$ then there is an $x$ in $A$ such that $f(x) = y$.

Def. The statement that the set $A$ is countable means there is a surjective $f : \mathbb{Z}^+ \to A$.

Problems

1. Find the limit point(s) of the following set: $C = \{n \cdot (2^5 \mod 1) : n \text{ is a positive integer}\}$
2. There exists a sequence $S$ such that $S$ has a limit and $\text{Im}(S)$ has exactly 2 limit points.
   (A direct proof)
3. If the sequence $S$ has a limit then the limit is unique.
4. l.p. if l.p.2 (l.p. is the open interval definition, l.p.2 uses distance)
5. There is a number set, $A$, which has the property that $A$ contains no open interval and each point in $A$ is a limit point of $A$.
6. If $A$ is an infinite subset of $[0,1]$ then $A$ has a limit point.
7. If $A$ is open and $p$ is in $A$ and there is a $q > p$ such that $q$ is not in $A$ then there is a $b$ not in $A$ such that $(p,b)$ is a subset of $A$.

Def. The statement that the number set $A$ has measure zero means that if $c > 0$ there is a countable (possibly finite) collection of open intervals $(a_i, b_i)$ such that $A \subseteq \bigcup_i (a_i, b_i)$ and

8. If $A$ is countable then $A$ has measure 0.

Axiom 2 If $A$ is a number set and $b$ is a number such that if $x \in A$ then $x \leq b$, then there is a number $q$ such that if $x$ is in $A$ then $x \leq q$ and if $r < q$ then there is a $y$ in $A$ such that $r < y \leq q$.

10. If $S$ is a monotonically non-decreasing number sequence which is bounded above, then $S$ has a limit.

Def If $[a,b]$ is a closed interval, then by a subdivision of $[a,b]$ we mean a finite collection $D$ of non-overlapping closed intervals whose union is $[a,b]$. (Two closed intervals are nonoverlapping provided either they do not intersect or their intersection is a single point.) An equivalent definition is that a subdivision of $[a,b]$ is a finite increasing sequence in $[a,b]$ such that its initial point is $a$ and is final point is $b$.

Def The statement that the subdivision $D$ refines the subdivision $D'$ means that if $d$ is in $D$ then there is an $e$ in $D'$ such that $d$ is a subset of $e$. In this case we say that $D$ is a refinement of $D'$. (Equivalently using the sequence definition, $D$ refines $D'$ means that $D'$ is a subsequence of $D$.)

Def If $D$ is a collection of sets, then a choice function for $D$ is a function $\text{ch}$ from $D$ into the union of $D$ with the property that $\text{ch}(d)$ is in $d$ for each $d$ in $D$. }
Def If $D$ is a subdivision of $[a,b]$, then the mesh of $D$ is $\max\{|d|:d \text{ is in } D\}$. ($|d|$ denotes the length of $d$.)

Def Suppose $[r,s]$ is a closed interval and $g$ is a real valued function whose domain contains $[r,s]$. The $g$-length of $[r,s]$ is the number $g(r)-g(s)$ and will be denoted by $g[r,s]$.

We are now in a position to define two (possibly different) notions of integral. Suppose $I=[a,b]$ and each of $f$ and $g$ is a real valued function defined on $I$.

Def The statement that $f$ is integrable on $I$ with respect to $g$ means there is a number $m$ such that if $\varepsilon > 0$ there is a $\delta > 0$ such that if $D$ is a subdivision of $I$ with mesh less than $\delta$ and $ch$ is a choice function for $D$, then

$$\sum_{d \in D} f(ch(d)) g |d - m| < \varepsilon.$$  

In this case we will denote the number $m$ by $\int_a^b f \ dg$.

Def The statement that $f$ is type-R integrable on $I$ with respect to $g$ means there is a number $m$ such that if $\varepsilon > 0$ there is a subdivision $D'$ of $I$ such that if $D$ refines $D'$ and $ch$ is a choice function for $D$ then

$$\sum_{d \in D} f(ch(d)) g |d - m| < \varepsilon.$$  

In this case we will denote the number $m$ by $R \int_a^b f \ dg$.

11. If $f$ is integrable on $I$ with respect to $g$ then the number $m$ is unique.

Def If $p$ is a point in $\mathbb{R}^2$ and epsilon is greater than zero, then the epsilon ball centered at $p$ is defined to be the set of points within epsilon of $p$. This is also referred to as the epsilon neighborhood of $p$. This definition naturally extends the topology of the line to that of the plane, and beyond. Of course, since we also have limit points defined in terms of distance, as long as we have the concept of distance, which we already do in Euclidean $n$-space, it is possible to mimic many of the theorems from 431 to higher dimensional spaces.

12. If $A$ is a subset of $\mathbb{R}^2$ that is infinite and bounded, then $A$ has a limit point.
13. If $f$ is continuous over $[a,b]$ and $f(a) < y < f(b)$, then there is a $c$ in $D$ such that $y = f(c)$.
14. If $f$ is continuous over the compact set $A$ and $c > 0$, then there is a $d > 0$ such that if each of $x$ and $y$ is in $A$ and $|x - y| < d$, the $|f(x) - f(y)| < c$.
15. If $S$ is a sequence which satisfies if $c>0$ there is a positive integer $N$ such that if $i,j > N$ then $|S(i) - S(j)|<c$, then $S$ has a limit.
16. $\int_a^b (g+h) = \int_a^b f \ dg + \int_a^b f \ dh$.
17. If $f$ is continuous over $U$ and $V$ is a subset of $U$ then $f$ restricted to $V$ is continuous.
18. a. If $f[0,1]\to\mathbb{R}$ is non-decreasing, then $f$ is B.V.
18b. If $f:[0,1]\to\mathbb{R}$ and $a \in (0,1)$ and each of $f_{[0,a]}$ and $f_{[a,1]}$ is B.V. then $f$ is B.V.

19. Show that $f(x) = \begin{cases} 0 & 0 \leq x < .5 \\ 1 & .5 \leq x \leq 1 \end{cases}$ is Riemann integrable over $[0,1]$.

20. Show that every point in the Cantor set is a limit point of the Cantor set.

21. The product of continuous functions is continuous over a common domain.
Math 431

We begin Math 431 by considering problems concerning the set, \( \mathbb{R} \), of real numbers. These numbers we consider to be in one-to-one correspondence with points on a line, ordered from left to right in the usual way. We will assume all the familiar arithmetic and order and thus, subsets of the numbers may be defined by statements involving arithmetic and order. For example:

\[ S = \{ x \colon x \text{ is in } \mathbb{R} \text{ and } 2 + x > 4 \} \]

is the set to which \( x \) belongs if and only if \( x \) is greater than 2.

**Def.** The number set, \( A \), is a **right ray** means that if \( x \) is in \( A \) and \( y > x \), then \( y \) is in \( A \). We similarly define a left ray.

In addition, we will assume that the following statement holds for the numbers. We call this statement an Axiom and emphasize that it is an **assumption** that does not follow from the usual properties of arithmetic and order. Nevertheless, it is a reasonable property for the points on a line to have and we wish to assume that the set of real numbers also has it.

**Axiom 1** If \( \mathbb{R} \) is the union of the nonempty left ray \( A \) and the nonempty right ray \( B \), and \( A \) and \( B \) do not intersect, then either \( A \) has a largest element or \( B \) has a smallest element.

If each of \( a \) and \( b \) is a number such that \( a < b \), then we will use the following notation and terminology.

- **open interval** \( (a, b) = \{ x \colon x \text{ is a number and } a < x \text{ and } x < b \} \)
- **half-open interval** \( [a, b) = \{ x \colon x \text{ is a number and } a < x \text{ and } x < b \} \)
- **half-open interval** \( (a, b] = \{ x \colon x \text{ is a number and } a < x \text{ and } x < b \} \)
- **closed interval** \( [a, b] = \{ x \colon x \text{ is a number and } a < x \text{ and } x < b \} \)

**Def.** The statement that the number, \( p \), is a **limit point** of the number set, \( A \), means that if \( (a, b) \) is an open interval containing \( p \), (that is: \( a < p < b \)) then there is a number \( q \) such that \( a < q < b \), \( q \) is in \( A \), and \( q \) does not equal \( p \).

A more concise statement of the previous definition may read: \( P \) is a limit point of \( A \) if and only if each open interval containing \( p \) contains a number in \( A \) different from \( p \).

**Def.** The statement that the number set, \( A \), is **closed** means that if \( p \) is a limit point of \( A \) then \( p \) is in \( A \).

**Def.** The statement that the number set, \( A \), is **open** means that if \( p \) is in \( A \) then there is an open interval containing \( p \) that is contained in \( A \).

**Def.** The statement that \( f \) is a **function** means that \( f \) is a collection, each member of which is an ordered pair, no two of which have the same first coordinate. The set of first coordinates for \( f \) is called the domain of \( f \), while the set of second coordinates is called the image of \( f \).

**Def.** The statement that \( S \) is a **sequence** means that \( S \) is a function with domain some initial segment of the positive integers. (That is: the domain of \( S \) is either the set of positive integers or the domain of \( f \) is the set \( \{1, 2, 3, ..., n\} \) for some positive integer \( n \)?)
**Def.** The statement that \( p \) is the limit of the sequence \( S \) means that if \((a,b)\) is an interval containing \( p \), then there is a positive integer \( N \) such that \( S(i) \) is in \((a,b)\) for each positive integer \( i \geq N \).

**Def.** The statement that \( T \) is a subsequence of the sequence \( S \) means there is an increasing sequence, \( I \), of positive integers such that \( T = S(I) \).

**Def.** The statement that the function \( f : A \to B \) is surjective (another name is “onto”) means that if \( y \) is in \( B \) then there is an \( x \) in \( A \) such that \( f(x) = y \).

**Def.** The statement that the set \( A \) is countable means there is a surjective \( f : \mathbb{Z}^+ \to A \).

**Problems**

1. Find the limit point(s) of the following set: \( C = \{ n \times (2.5)^k \mod 1 : n \text{ is a positive integer} \} \)
2. There exists a sequence \( S \) such that \( S \) has a limit and \( \text{Im}(S) \) has exactly 2 limit points. (A direct proof)
3. If the sequence \( S \) has a limit then the limit is unique.
4. l.p. iff l.p.2 (l.p. is the open interval definition, l.p.2 uses distance)
5. There is a number set, \( A \), which has the property that \( A \) contains no open interval and each point in \( A \) is a limit point of \( A \).
6. If \( A \) is an infinite subset of \([0,1]\) then \( A \) has a limit point.
7. If \( A \) is open and \( p \) is in \( A \) and there is a \( q > p \) such that \( q \) is not in \( A \) then there is a \( b \) not in \( A \) such that \( (p,b) \) is a subset of \( A \).

**Def.** The statement that the number set \( A \) has measure zero means that if \( c > 0 \) there is a countable (possibly finite) collection of open intervals \( (a_i, b_i) \) such that \( A \subseteq \bigcup_i (a_i, b_i) \) and

8. If \( A \) is countable then \( A \) has measure 0.

**Axiom 2** If \( A \) is a number set and \( b \) is a number such that if \( x \in A \) then \( x \leq b \), then there is a number \( q \) such that if \( x \) is in \( A \) then \( x \leq q \) and if \( r < q \) then there is a \( y \) in \( A \) such that \( r < y \leq q \).

10. If \( S \) is a monotonically non-decreasing number sequence which is bounded above, then \( S \) has a limit.

**Def** If \([a,b]\) is a closed interval, then by a subdivision of \([a,b]\) we mean a finite collection \( D \) of non-overlapping closed intervals whose union is \([a,b]\). (Two closed intervals are nonoverlapping provided either they do not intersect or their intersection is a single point.) An equivalent definition is that a subdivision of \([a,b]\) is a finite increasing sequence in \([a,b]\) such that its initial point is \( a \) and is final point is \( b \).

**Def** The statement that the subdivision \( D \) refines the subdivision \( D' \) means that if \( d \) is in \( D \) then there is an \( e \) in \( D' \) such that \( d \) is a subset of \( e \). In this case we say that \( D \) is a refinement of \( D' \). (Equivalently using the sequence definition, \( D \) refines \( D' \) means that \( D' \) is a subsequence of \( D \).)

**Def** If \( D \) is a collection of sets, then a choice function for \( D \) is a function \( ch \) from \( D \) into the union of \( D \) with the property that \( ch(d) \) is in \( d \) for each \( d \) in \( D \).
Def If D is a subdivision of \([a,b]\), then the mesh of D is \(\max\{|d|: d \text{ is in } D\}\). \(|d|\) denotes the length of d.)

Def Suppose \([r,s]\) is a closed interval and g is a real valued function whose domain contains \([r,s]\). the g-length of \([r,s]\) is the number \(g(r)-g(s)\) and will be denoted by \(g[r,s]\).

We are now in a position to define two (possibly different) notions of integral. Suppose \(I=[a,b]\) and each of f and g is a real valued function defined on I.

Def The statement that \(f\) is integrable on I with respect to g means there is a number m such that if \(\varepsilon > 0\) there is a \(\delta > 0\) such that if D is a subdivision of I with mesh less than \(\delta\) and ch is a choice function for D, then
\[
\left| \sum_{d \in D} f(ch(d)) g(d) - m \right| < \varepsilon.
\]
In this case we will denote the number m by
\[
\int_a^b f \, dg.
\]

Def The statement that \(f\) is type-R integrable on I with respect to g means there is a number m such that if \(\varepsilon > 0\) there is a subdivision D' of I such that if D refines D' and ch is a choice function for D then
\[
\left| \sum_{d \in D} f(ch(d)) g(d) - m \right| < \varepsilon.
\]
In this case we will denote the number m by
\[
\int_a^b f \, dg.
\]

11. If \(f\) is integrable on I with respect to g then the number m is unique.

Def If p is a point in \(\mathbb{R}^2\) and epsilon is greater than zero, then the epsilon ball centered at p is defined to be the set of points within epsilon of p. This is also referred to as the epsilon neighborhood of p. This definition naturally extends the topology of the line to that of the plane, and beyond. Of course, since we also have limit points defined in terms of distance, as long as we have the concept of distance, which we already do in Euclidean n-space, it is possible to mimic many of the theorems from 431 to higher dimensional spaces.

12. If A is a subset of \(\mathbb{R}^2\) that is infinite and bounded, then A has a limit point.
13. If \(f\) is continuous over \([a,b]\) and \(f(a) < y < f(b)\), then there is a \(c\) in D such that \(y = f(c)\).
14. If \(f\) is continuous over the compact set A and \(c > 0\), then there is a \(d > 0\) such that if each of x and y is in A and \(|x - y| < d\), the \(|f(x) - f(y)| < c\).
15. If S is a sequence which satisfies if \(c > 0\) there is a positive integer N such that if i, j > N then \(|S(i) - S(j)| < c\), then S has a limit.
16. \[
\int_a^b f \, (g + h) = \int_a^b f \, dg + \int_a^b f \, dh.
\]
17. If \(f\) is continuous over U and V is a subset of U then \(f\) restricted to V is continuous.
18. a. If \(f[0,1] \rightarrow \mathbb{R}\) is non-decreasing, then \(f\) is B.V.
18b. If $f:[0,1] \rightarrow \mathbb{R}$ and $a \in (0,1)$ and each of $f_{[0,a]}$ and $f_{[a,1]}$ is B.V. then $f$ is B.V.

19. Show $f(x) = \begin{cases} 0 & 0 \leq x < .5 \\ 1 & .5 \leq x \leq 1 \end{cases}$ is Riemann integrable over $[0,1]$.

20. Show that every point in the Cantor set is a limit point of the Cantor set.

21. The product of continuous functions is continuous over a common domain.

Math 431-432 Portfolio Assessment Rubric

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