PHYS331: Optics

Equations & Constants

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Constants

- Speed of EM wave in vacuum: $c = 2.998 \times 10^5\,\mathrm{km\,s^{-1}} = \dfrac{1}{\sqrt{\varepsilon_0 \mu_0}}$
- Speed of EM wave in medium: $v = \dfrac{1}{\sqrt{\varepsilon \mu}}$
- Permittivity of vacuum: $\varepsilon_0 = 8.8542 \times 10^{-12}\,\mathrm{C^2\,s^2\,kg^{-1}\,m^{-3}}$
- Permeability of vacuum: $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{kg\,m\,A^{-2}\,s^{-2}}$
- Wien's Displacement Law constant: $\kappa = 2.898 \times 10^6\,\mathrm{nm\,K}$
- Stefan-Boltzmann constant: $\sigma = 5.67 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$
- Planck constant: $h = 6.626 \times 10^{-34}\,\mathrm{J\,s}$
- Boltzmann constant: $k_\mathrm{B} = 1.381 \times 10^{-23}\,\mathrm{J\,K^{-1}}$
- Coulomb constant: $k_\mathrm{C} = 9 \times 10^9\,\mathrm{N\,m^2\,C^{-2}}$
- Solar luminosity: $1\,\mathrm{L_\odot} = 3.83 \times 10^{26}\,\mathrm{W}$
- Electron-volt: $1\,\mathrm{eV} = 1.602 \times 10^{-19}\,\mathrm{J}$
- Ampere: $1\,\mathrm{A} = 1\,\mathrm{C\,s^{-1}}$
- Volt: $1\,\mathrm{V} = 1\,\mathrm{kg\,m^2\,A^{-1}\,s^{-3}}$
- Tesla: $1\,\mathrm{T} = 1\,\mathrm{V\,s\,m^{-2}}$
- Farad: $1\,\mathrm{F} = 1\,\mathrm{C\,V^{-1}}$
- Diopter $= \dfrac{1\,\mathrm{m}}{f}$, where f is focal length.

Indices of Refraction $c = nv$	
Vacuum	1
Air	1.000277
Water	1.3330
Human eye	1.34
Pyrex	1.47
Crown glass	1.50
Flint glass	1.60
Diamond	2.419

Fraunhofer Lines			
		n	
λ (nm)	Characterization	Crown glass	Flint glass
486.1	F, blue	1.5286	1.7328
589.2	D, yellow	1.5230	1.7205
656.3	C, red	1.5205	1.7076

Equations

- Wave-particle duality
  - Momentum of particle: $p = \dfrac{\sqrt{E^2 - (mc^2)^2}}{c}$
  - Wavelength of particle: $\lambda = \dfrac{h}{p} = \dfrac{hc}{\sqrt{E^2 - (mc^2)^2}}$
  - Speed of particle: $v = \dfrac{pc^2}{E} = c\sqrt{E^2 - (mc^2)^2}$
- Wave equations:
  - Generic 3D wave equation: $\psi = Ae^{i(\vec{k} \bullet \vec{r} - \omega t)}$, where physical wave is real or imaginary component.
  - Propagation constant: $k = \dfrac{2\pi}{\lambda}$
  - Angular frequency: $\omega = 2\pi\nu$
  - Harmonic condition: $\nabla^2 \psi = \dfrac{1}{v^2}\dfrac{\partial^2 \psi}{\partial t^2}$
- Superposition of N harmonic (plane) waves, $E_i = E_{0i}\cos(\alpha_i - \omega t)$:
  - Resulting constant phase:
  $$\tan\alpha = \frac{\sum_{i=1}^{N} E_{0i}\sin\alpha_i}{\sum_{i=1}^{N} E_{0i}\cos\alpha_i}.$$
  - Resulting amplitude:
  $$E_0^2 = \sum_{i=1}^{N} E_{0i}^2 + 2\sum_{i=1}^{N}\sum_{j=i+1}^{N} E_{0i}E_{0j}\cos(\alpha_j - \alpha_i)$$
- Electromagnetic (EM) waves:
  - Electric and magnetic field relation: $|\vec{\mathrm{E}}| = v|\vec{B}|$, where v is speed of EM wave.
  - Total energy density of EM wave: $u = \varepsilon E^2 = \dfrac{1}{\mu}B^2$
  - Poynting vector: $\vec{\mathrm{S}} = \dfrac{\varepsilon c^2}{n^2}\vec{\mathrm{E}} \times \vec{\mathrm{B}}$ (power per unit area of EM field).
  - Irradiance: $E_e = \langle|\vec{\mathrm{S}}|\rangle = \frac{1}{2}\varepsilon_0 c^2 E_0 B_0$

Radiometry			
Term	**Symbol (units)**	**Defining equation**	**AKA**
Radiant energy	Q_e (J = W s)	...	Energy
Radiant energy density	w_e ($\mathrm{J\,m^{-3}}$)	$w_e = dQ_e/dV$	Energy density
Radiant flux, Radiant power	Φ_e (W)	$\Phi_e = dQ_e/dt$	(Bolometric) Luminosity
Radiant flux density			
Emitted: Radiant exitance	M_e ($\mathrm{W\,m^{-2}}$)	$M_e = d\Phi_e/dA$	Intensity, Flux
Incident: Irradiance	E_e ($\mathrm{W\,m^{-2}}$)	$E_e = d\Phi_e/dA$	Intensity, Flux
Radiant intensity	I_e ($\mathrm{W\,sr^{-1}}$)	$I_e = d\Phi_e/d\omega$	...
Radiance	L_e ($\mathrm{W\,sr^{-1}\,m^{-2}}$)	$L_e = dI_e/(dA\cos\theta)$	...

- Fourier series:
  $f(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos m\omega t + \sum_{n=1}^{\infty} b_m \sin m\omega t$
    * $a_0 = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)dt$
    * $a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos n\omega t dt$
    * $b_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin n\omega t dt$
  - where T is a period of $f(t)$.
- Fourier-transform pair:
    * $f(t) = \int_{-\infty}^{+\infty} g(\omega)e^{-i\omega t}d\omega$
    * $g(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{i\omega t}dt$
- Complex numbers:
  - Generic complex number: $\tilde{z} = a+ib$ where
    * $a = \mathrm{Re}(\tilde{z})$,
    * $b = \mathrm{Im}(\tilde{z})$, and
    * $|\tilde{z}|^2 = a^2 + b^2$.
  - Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$, where $\theta = \tan^{-1}\left(\frac{b}{a}\right)$ for $\tilde{z}$ above.
  - Complex conjugate: $\tilde{z}^* = a - ib = |\tilde{z}|e^{-i\theta}$
- Snell's law: $n_1 \sin\theta_1 = n_2 \sin\theta_2$
- Focal length of spherical mirror: $f = -\frac{R}{2}$
- Mirror equation and thin-lens equation: $\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}$
- Total magnification: $M = \prod_i m_i$
- Lateral magnification: $m = \frac{h'}{h} = -\frac{n_1 s'}{n_2 s}$
- Angular magnification: $m = \frac{\alpha'}{\alpha} = -\frac{f_\mathrm{o}}{f_\mathrm{e}}$
- Angular magnification of simple magnifier:
    * $M = \frac{25\,\mathrm{cm}}{f}$ $(s' = \infty)$
    * $M = \frac{25\,\mathrm{cm}}{f} + 1$ $(s' = -25\,\mathrm{cm})$
- Refraction at spherical surface: $\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{R}$
- Lensmaker's equation: $\frac{1}{f} = \frac{n_2 - n_1}{n_1}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$
- Refracting power: $P = V + V'$, where vergence $V = \frac{1}{s}$ and $V' = \frac{1}{s'}$.
- Thin lenses in contact have effective focal lengths: $\frac{1}{f_\mathrm{eff}} = \sum_i \frac{1}{f_i}$
- Two thin lenses separated by length L have effective focal length (from principal plane 1): $\frac{1}{f_\mathrm{eff}} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{L}{f_1 f_2}$
- Newtonian equation for thin lens: $xx' = f^2$, where x is distance from object to focal point on object side and x' is distance from focal point on other side to image.
- Image length of cylindrical lens: $\overline{AB} = \left(\frac{s+s'}{s}\right)\overline{CL}$
- Thick lens equations:
    * $\frac{1}{f_1} = \frac{n_\mathrm{L} - n'}{nR_2} - \frac{n_\mathrm{L} - n}{nR_1} - \frac{(n_\mathrm{L} - n)(n_\mathrm{L} - n')}{nn_\mathrm{L}}\frac{t}{R_1 R_2}$
    * $f_2 = -\frac{n'}{n} f_1$
    * Principal point 1: $r = \frac{n_\mathrm{L} - n'}{n_\mathrm{L} R_2} f_1 t$
    * Principal point 2: $s = -\frac{n_\mathrm{L} - n}{n_\mathrm{L} R_1} f_2 t$
    * Nodal point 1: $v = \left(1 - \frac{n'}{n} + \frac{n_\mathrm{L} - n'}{n_\mathrm{L} R_2} t\right) f_1$
    * Nodal point 2: $w = \left(1 - \frac{n}{n'} - \frac{n_\mathrm{L} - n}{n_\mathrm{L} R_1} t\right) f_2$
    * Object-image distance; $-\frac{f_1}{s_\mathrm{o}} + \frac{f_2}{s_\mathrm{i}} = 1$
    * Lateral magnification: $m = -\frac{s_\mathrm{i}}{s_\mathrm{o}}$
- $\mathrm{Det}\,[\mathcal{M}_\mathrm{sys}] = \left|\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right| = AD - BC = \frac{n_0}{n_f}$
- Simple ray-transfer matrices:
  - Translation: $\mathcal{T} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}$
  - Refraction (spherical): $\mathcal{R}_\mathrm{sph} = \begin{bmatrix} 1 & 0 \\ \frac{n - n'}{n'R} & \frac{n}{n'} \end{bmatrix}$
  - Thin-lens: $\mathcal{L} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$
  - Spherical mirror: $\mathcal{S} = \begin{bmatrix} 1 & 0 \\ \frac{2}{R} & 1 \end{bmatrix}$
- Cardinal point locations in terms of system matrix elements (with respect to (w.r.t.) input (1) and output (2) reference planes unless noted):
  - F_1:
    * $p = \frac{D}{C}$
    * $f_1 = p - r = \frac{\frac{n_0}{n_f}}{C}$ (w.r.t. prin. plane)
  - F_2:
    * $q = -\frac{A}{C}$
    * $f_2 = q - s = -\frac{1}{C}$ (w.r.t. prin. plane)
  - H_1: $r = \frac{D - \frac{n_0}{n_f}}{C}$
  - H_2: $s = \frac{1 - A}{C}$
  - N_1: $v = \frac{D - 1}{C}$
  - N_2: $w = \frac{\frac{n_0}{n_f} - A}{C}$

- Relative aperture (AKA f-number, f/stop): $A = \frac{f}{D}$
- Depth-of-field: $\mathrm{DOF} = \frac{2Ads_0(s_0 - f)f^2}{f^4 - A^2 d^2 s_0^2}$, where s_0 is object distance.
- Near-point: $s_1 = \frac{s_0 f(f + Ad)}{f^2 + Ads_0}$
- Far-point: $s_2 = \frac{s_0 f(f - Ad)}{f^2 - Ads_0}$
- Cartesian ovoid of revolution: constant $= n_o(x^2 + y^2)^{1/2} + n_i[y^2 + (s_o + s_i - x)^2]^{1/2}$
- Index of refraction of prism: $$n = \frac{\sin\left(\frac{A+\delta}{2}\right)}{\sin\left(\frac{A}{2}\right)}$$
- Minimum deviation of a prism: $\delta \approx A(n-1)$
- Normal dispersive curve: $n_\lambda = A + \frac{B}{\lambda^2} + \frac{C}{\lambda^4} + ...$, where $A, B, C, ...$ are empirical (Cauchy approximation).
- Dispersion: $\mathcal{D} = \frac{n_\lambda}{\lambda}$
- Dispersive power: $\Delta \equiv \frac{\mathcal{D}}{\delta} = \frac{n_F - n_C}{n_D - 1}$
- Resolving power: $\mathcal{R} \equiv \frac{\lambda}{(\Delta\lambda)_{\min}} = b\frac{dn_\lambda}{d\lambda}$
- Miscellaneous Physics:
  - Wien's Displacement Law: $\lambda_{\text{peak}} = \kappa T^{-1}$
  - Stefan-Boltzmann Law: $j = \sigma T^4$, where j is flux at surface.
  - Luminosity-flux relation: $L = A\,F$, where A is area.
  - Pressure: $P = \frac{F}{A}$, where F is force and A is area.
  - Radiation pressure: $P = \frac{F}{c}$, where F is flux.
  - Average kinetic (i.e., motion) energy of particles: $E \approx k_{\mathrm{B}}\,T$,
  - Ideal gas law (gas pressure): $P = n\,k_{\mathrm{B}}\,T$, where n is particle number density.
- Quadratic solution for $ax^2 + bx + c = 0$: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- Geometry:
  - Solid angle between two points (in steradians) is $\Omega = \frac{A}{r^2}$, where A is area subtended at distance r.
  - Surface area of sphere: $A = 4\pi r^2$
  - Volume of sphere: $V = \frac{4}{3}\pi r^3$
  - Volume of a cylinder: $V = \pi r^2 h$
- Cross product: $\overrightarrow{\mathrm{A}} \times \overrightarrow{\mathrm{B}} = AB\sin\theta$
- Dot product: $\overrightarrow{\mathrm{A}} \bullet \overrightarrow{\mathrm{B}} = AB\cos\theta$
- Laplacian operator: $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$
- Spherical coordinates:
  - $\hat{\boldsymbol{r}} = \sin\theta\cos\phi\,\hat{\boldsymbol{x}} + \sin\theta\sin\phi\,\hat{\boldsymbol{y}} + \cos\theta\,\hat{\boldsymbol{z}}$
 $r = \sqrt{x^2 + y^2 + z^2}$
  - $\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\boldsymbol{x}} + \cos\theta\sin\phi\,\hat{\boldsymbol{y}} - \sin\theta\,\hat{\boldsymbol{z}}$
 $\theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$
  - $\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\boldsymbol{x}} + \cos\phi\,\hat{\boldsymbol{y}}$
 $\phi = \tan^{-1}\left(\frac{y}{x}\right)$
- Cylindrical coordinates:
  - $\hat{\boldsymbol{s}} = \cos\phi\,\hat{\boldsymbol{x}} + \sin\phi\,\hat{\boldsymbol{y}}$
 $s = \sqrt{x^2 + y^2}$
  - $\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\boldsymbol{x}} + \cos\phi\,\hat{\boldsymbol{y}}$
 $\phi = \tan^{-1}\left(\frac{y}{x}\right)$
  - $\hat{\boldsymbol{z}} = \hat{\boldsymbol{z}}$
 $z = z$
- Taylor series: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$, where $f^{(n)}(a)$ denotes nth derivative of f evaluated at point a. Factorial of zero is one (i.e., $0! = 1$).
  - $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + ... = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\theta^{2n+1}$
  - $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + ... = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\theta^{2n}$
  - $\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \frac{17\theta^7}{315} + ...$, for $|\theta| < \frac{\pi}{2}$.
- Useful trigonometric equations:
  - $\int \sin x dx = -\cos x$
  - $\int \cos x dx = \sin x$
  - $\sin(A \pm B) = \sin A\cos B \pm \sin B\cos A$
  - $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$
  - $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$; $\cos\frac{\pi}{3} = \frac{1}{2}$
  - $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$; $\cos\frac{2\pi}{3} = -\frac{1}{2}$
  - $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$
  - $\sin\frac{3\pi}{4} = \frac{\sqrt{2}}{2}$; $\cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$
  - $\sin\left(n\pi + \frac{\pi}{2}\right) = \cos n\pi = (-1)^n$ where integer $n \geq 0$

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